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Abstract

We investigate a Stackelberg oligopoly model in which m leaders and $N - m$ followers compete. We find an asymmetric welfare implication of the Stackelberg model. Introducing a small number of leaders into the Cournot model can reduce welfare. However, introducing a small number of followers into the Cournot model always improves welfare. The key result behind this asymmetry is contrasting limit results in the cases where $m \to 0$ and $m \to N$. We also discuss the optimal number of leaders and the integer constraint for the number of the firms.

JEL classification numbers: L13, L40

Key words: multiple leaders, Stackelberg, Cournot, limit result, integer constraint, convex cost

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1 Introduction

Cournot and Stackelberg models are important models in the literature concerning imperfect competition, and the welfare implications of these models are intensively discussed in the literature. Many works have considered Stackelberg duopolies in which one firm, called the Stackelberg leader, competes against the other, called the Stackelberg follower. However, duopoly markets are rare, and it is important to generalize the welfare analysis in the multi-leader and multi-follower cases.

Daughety (1990) and Ohkawa and Okamura (2003) formulate a model in which m Stackelberg leaders and $N - m$ Stackelberg followers compete in a homogeneous goods market[1] and all firms are identical in all respects except for their roles (either a leader or a follower). Using a linear demand function and a constant marginal cost, these works identify an inverse U-shaped relationship between m and economic welfare: welfare is minimized when $m = 0$ and $m = N$, where both conditions correspond to the Cournot model. The inverse U-shaped relationship between m and economic welfare indicates that either introducing a small number of leaders or a small number of followers improves welfare.[2]

In this paper, we show that introducing a small number of *leaders* into the Cournot model does not always improve welfare. However, it is shown that introducing a small number of *followers* into the Cournot model always improves welfare. These contrasting results indicate that the existence of a small number of followers has quite different implications from those of the existence of a small number of leaders.[3] For instance, it might be reasonable that antitrust departments focus on markets with a small number of leaders but do not care about markets with a small number of followers.

Our results also contain implications on the relationship between HHI (Herfindahl-Hirschman

[1] Sherali (1984) provided the same kind of multiple-leader Stackelberg model earlier though the welfare analysis is not this papaer's interest.

[2] Ino and Matsumura (2012) consider the free-entry multiple-leader Stackelberg models by endogenizing the number of followers and show that this result (introducing leaders into a Cournot model always improves welfare) holds in more general setting.

[3] We think that the result of a small number of followers is important because it is the equilibrium outcome in an endogenous timing game. See Matsumura (1999). Pal (1998) also presents a result in which only one firm (public firm) becomes the follower in a mixed oligopoly.

Index) and economic welfare. Suppose that all firms are identical except for the timing of production. Stackelberg models yield a higher HHI, which indicates a higher degree of concentration of the market, than the Cournot model. Daughety's result suggests that the Stackelberg models always yields a larger welfare than the Cournot model regardless of the number of the leaders. Thus, beneficial concentration always occurs. Our results suggest that some Stackelberg models (in which the number of the leaders is small) yields a smaller welfare than the Cournot model, but there always exists a Stackelberg model (in which the number of the followers is small) yielding a larger welfare. Thus, beneficial concentration can occur and this result is robust in a sense that it does not depend on the linear demand or the constant marginal cost.

The rest of paper is organized as follows. In Section 2, we formulate the model. In Section 3, we provide our main results. In Section 4, we further elaborate on the relationship between welfare and the number of leaders, such as the optimal number of leaders and the integer constraint of the number of firms by specifying the demand and cost functions. All proofs are relegated to the Appendix.

2 The Model

We formulate an N-firm oligopoly model ($N > 0$). We neglect the integer problems on the number of firms.[4] Firms produce a homogeneous product with an identical cost function, $C(x) : \mathbb{R}_+ \mapsto \mathbb{R}_+$. The (inverse) demand function is given by $P(X) : \mathbb{R}_+ \mapsto \mathbb{R}_+$. Each firm's payoff is its own profit. Firm i's profit is given by $P(X)x_i - C(x_i)$, where x_i is firm i's output and $X \equiv \sum_{i=1}^{N} x_i$. We make the following standard assumptions.

Assumption 1. $P(X)$ is twice differentiable and $P'(X) < 0$ for all X such that $P(X) > 0$.

Assumption 2. $C(x)$ is twice differentiable, $C'(x) \geq 0$ for all $x \geq 0$, and $P'(X) - C''(x) < 0$ for all

[4] As well as this paper, marginal effects of the number of firms are often discussed in order to grasp the essential tendencies of oligopoly models. For instance in the literature, see Mankiw and Whinston (1986). They provide the well-known "excess entry" theorem which implies that a *marginal* reduction in the number of firms improves welfare. Needless to say, the integer problems are also worth to discuss. If we restrict our attention to the linear demand and the quadratic production cost, even under the integer constraint $N \in \{2, 3, \cdots\}$, we obtain an asymmetric welfare implication similar to our main result. See Section 4.3.

X such that $P(X) > 0$ and for all $x \in (0, X)$.

Assumption 3. $P''(X)x + P'(X) < 0$ for all X such that $P(X) > 0$ and for all $x \in (0, X)$.

Assumption 1 is the standard assumption that the demand curve is downward-sloping. Assumption 2 is satisfied if (but not only if) $C' \geq 0$ and $C'' \geq 0$ under Assumption 1. Assumption 3 ensures that each firm's marginal revenue is decreasing in the rivals' outputs and that the reaction curve in the Cournot model has a negative slope (strategic substitute).

The game runs as follows. In the first stage, $m(\in [0, N])$ firms (Stackelberg leaders) independently choose their outputs. In the second stage, n $(\equiv N - m)$ firms (Stackelberg followers) independently choose their outputs after observing the leaders' outputs. When $m = 0$ or $m = N$, the model corresponds to the Cournot model, in which N firms produce simultaneously and independently.[5]

We call the subgame perfect equilibrium symmetric if all leaders (followers) choose the same output level. Needless to say, a leader's output is different from a follower's even in the symmetric equilibrium. We focus on this symmetric equilibrium. We assume that the symmetric equilibrium uniquely exists[6] for all $m \in [0, N]$ and that both leaders and followers produce positive outputs.

3 Welfare

The solution of the following system of equations, $(x_L^*(m), x_F^*(m))$, corresponds to the equilibrium outputs of leaders and followers.[7] Note that (1) and (2) are the first-order conditions for leaders and

[5] In Proposition 1, we will show that each follower's output converges to each firm's output in the Cournot model when $m \to 0$ and each leader's converges when $m \to N$.

[6] Sherali (1984) provides a sufficient condition for the existence of a unique symmetric equilibrium in a model with multiple Stackelberg leaders. Specification given in the next section (linear demand and quadratic cost) satisfies Sherali's condition.

[7] More precisely, correspondence between $(x_L^*(m), x_F^*(m))$ and the equilibrium output is as follows. When $m \in (0, N)$, $x_L^*(m)$ represents a leader's equilibrium output and $x_F^*(m)$ represents a follower's (Stackelberg model). When $m = 0$ and $m = N$, (2) and (1) respectively correspond to the first-order condition of the Cournot model with N firms. Thus, $x_F^*(0) = x_L^*(N)$ is the equilibrium output of each firm when N firms produce simultaneously (Cournot model). Furthermore, though (1) ((2)) have no economic meaning when $m = 0$ $(m = N)$, we can solve this equation and obtain $x_L^*(0)$ $(x_F^*(N))$. We can show that (1) ((2)) have a solution when $m = 0$ $(m = N)$ and that $x_L^*(0)$ $(x_F^*(N))$ is well-defined.

followers, respectively:

$$(1 + nR'(X_L^*))P'(X^*)x_L^* + P(X^*) - C'(x_L^*) = 0, \tag{1}$$

$$P'(X^*)x_F^* + P(X^*) - C'(x_F^*) = 0, \tag{2}$$

where $X^*(m) = mx_L^*(m) + nx_F^*(m)$ and $X_L^*(m) = mx_L^*(m)$. $R(X_L)$ represents a follower's reaction to the leaders' total output X_L.[8] $R(X_L)$ is obtained from the follower's first-order condition

$$P'(X_L + nR(X_L))R(X_L) + P(X_L + nR(X_L)) - C'(R(X_L)) = 0. \tag{3}$$

Differentiating it yields

$$R'(X_L) = -\frac{P'(X_L + nR) + P''(X_L + nR)R}{n(P'(X_L + nR) + P''(X_L + nR)R) + (P'(X_L + nR) - C''(R))}. \tag{4}$$

From Assumptions 1–3, we have $nR'(X_L) \in (-1, 0)$ when $m \in [0, N)$, implying that an increase in the leaders' outputs decreases the followers' outputs and increases the total output of firms.

The equilibrium social welfare (consumer surplus plus profits of all firms) W^* is given by

$$W^*(m) = \int_0^{X^*} P(q)dq - mC(x_L^*) - nC(x_F^*). \tag{5}$$

Further, let the equilibrium profit of a leader be $\pi_L^*(m) = P(X^*(m))x_L^*(m) - C(x_L^*(m))$ and that of a follower be $\pi_F^*(m) = P(X^*(m))x_F^*(m) - C(x_F^*(m))$.

Let subscript "C" denote the equilibrium outcomes in the Cournot model. We present a well-known result highlighting the difference between the Cournot outcome (i.e., $m = 0$ or $m = N$) and the Stackelberg outcome (i.e., $m \in (0, N)$).

Lemma 1 *Suppose that Assumptions 1–3 are satisfied. Then, for all* $m \in (0, N)$, *(i)* $x_L^*(m) > x_C^*$, *(ii)* $x_F^*(m) < x_C^*$, *and (iii)* $X^*(m) > X_C^*$.

[8] Under Assumptions 1–3, we can show that all subgames in the second stage have a unique equilibrium that is symmetric.

Suppose that some leaders or followers are introduced into the Cournot model. The leader's output is strictly larger than the follower's (Lemma 1(i,ii)). When the marginal cost is increasing, total production costs are minimized when all firms produce the same output. The introduction of a leader (followers) yields a difference in the firms' production levels and thus reduces production efficiency.[9] On the other hand, the introduction of leaders (followers) increases total output and consumer surplus (Lemma 1(iii)). This output-expansion effect improves welfare. In general, the former welfare-reducing effect may or may not dominate the latter welfare-improving effect.[10] Thus, the welfare effect of introducing some number of leaders or followers into the Cournot model is ambiguous.

However, an interesting asymmetric welfare implication can be obtained if the number of leaders and followers introduced into the Cournot model is small. Regarding this, we now present our main results (Propositions 1–2). Proposition 1 presents an important limit property that helps us understand the asymmetric welfare implication presented in Proposition 2. Proposition 2 is the result indicating that while (i) introducing a small number of leaders into the Cournot model does not always improve welfare, (ii) introducing a small number of followers into the Cournot model always improves welfare.[11]

Proposition 1 *Suppose that Assumptions 1–3 are satisfied. Then,*
(i) $x_C^* = x_L^*(N) = \lim_{m\to N} x_L^*(m) = \lim_{m\to N} x_F^*(m) = x_F^*(N)$,
and (ii) $x_L^*(0) = \lim_{m\to 0} x_L^*(m) > \lim_{m\to 0} x_F^*(m) = x_F^*(0) = x_C^*$.

Proposition 2 *Suppose that Assumptions 1–3 are satisfied. Then, (i) we can have both the case where* $\frac{\partial W^*}{\partial m}\big|_{m=0} < 0$ *and the case where* $\frac{\partial W^*}{\partial m}\big|_{m=0} > 0$*, and (ii) we always have* $\frac{\partial W^*}{\partial m}\big|_{m=N} < 0$.

Proposition 1 states that both the Stackelberg leader's output and the follower's output converge

[9]Note that this effect does not exist when the marginal cost is constant.

[10]A similar trade-off between the two effects has been discussed in many contexts. See, among others, Lahiri and Ono (1988, 1998).

[11]Note that "introducing a small number of followers" implies that m *decrease* marginally from N. Needless to say, $\partial W^*/\partial m|_{m=N}$ represents how W^* moves when m *increase* marginally from N. Thus, the introduction of a small number of followers enhances welfare if the sign of $\partial W^*/\partial m|_{m=N}$ is negative.

to the Cournot output when m is close to N, whereas the leader's output does not converge to the Cournot output (the follower's output converges to the Cournot one) when m is close to 0.

We explain the intuition behind this asymmetry. When $m = N$, no follower exists. Thus, leaders do not take a strategic action against the other firms. Introducing a small number of followers into the Cournot model makes the leader take such a strategic action. However, the number of leaders is large whereas the number of followers is small. Thus, the strategic effect is small, and each leader expands its output slightly. This is why $x_L^*(m)$ is close to x_C^* when m is close to N. The same principle cannot be applied to the case where m is close to 0. Introducing a small number of leaders into the Cournot model makes the leader take a strategic action. Because the number of leaders is small and the number of followers is large, the strategic effect affects each leader's output significantly, resulting in a non-negligible difference between $x_L^*(m)$ and x_C^*. This is why $x_L^*(m)$ does not converge to x_C^* when m is close to 0.

We now explain the intuition behind the asymmetry between (i) and (ii) in Proposition 2. Suppose that a small number of followers are introduced into the Cournot model. As Proposition 1(i) indicates, both the leader's output and the follower's output converge to the Cournot output when m is close to N. Thus, introducing a small number of leaders does not yield a difference in the firms' production levels, and therefore, the former welfare-reducing effect discussed above (production-inefficiency effect) disappears and only the latter welfare-improving effect (output-expansion effect) remains.[12] As a result, introducing a small number of followers definitely improves welfare. On the other hand, suppose that a small number of leaders are introduced into the Cournot model. As seen in Proposition 1(ii), the difference between the leader's output and the follower's output is not negligible even when m is close to 0. Thus, the trade-off between production-inefficiency and output-expansion effects still remains. Introducing a small number of leaders reduces welfare when the former effect dominates the

[12]The output-expansion effect does not converge to zero even when $m \to N$. This is because the output expansion in this case occurs because the new followers (a marginal reduction of m) augment rivals' businesses. Because $R' < 0$ under the stability condition (assumption 3), a marginal decrease in output by the new followers *strictly* increases the rivals' outputs. Therefore, the marginal output expansion by augmenting the rivals' business is *strictly* positive, i.e., $-N\partial x_L^*(N)/\partial m > 0$.

latter effect. This is why introducing a small number of followers and a small number of leaders have contrasting welfare implications, as is shown in Proposition 2.

We provide another intuitive explanation[13] for Proposition 2. The following decomposition helps us to understand Proposition 2(i):

$$\frac{\partial W^*}{\partial m}\Big|_{m=0} = [\pi_L^*(0) - \pi_F^*(0)] + N[P(X^*(0)) - C'(x_F^*(0))]\frac{\partial x_F^*}{\partial m}\Big|_{m=0}. \tag{6}$$

The change in welfare due to the introduction of small number of leaders is decomposed into two parts. The first term is the direct effect of increasing the profit of a firm that changes the role (from the follower to the leader). Proposition 1(ii) implies that this effect is positive (the first-mover advantage remains significant even when $m \to 0$). At the same time, the new leader steals the other firms' business, resulting in the reduction of the other firms' (followers') outputs. Because the price is higher than the marginal cost, a welfare change induced by this indirect effect (second term) is negative. The change in welfare can be either negative or positive, depending on the relative size of these two effects.

Proposition 2(ii) is elucidated by the decomposition below.

$$\frac{\partial W^*}{\partial m}\Big|_{m=N} = [\pi_L^*(N) - \pi_F^*(N)] + N[P(X^*(N)) - C'(x_L^*(N))]\frac{\partial x_L^*}{\partial m}\Big|_{m=N}. \tag{7}$$

Similarly, the change in welfare by the introduction of a small number of followers is decomposed into two parts. In contrast to the previous case, Proposition 1(ii) implies that the first term is zero; that is, the first direct effect (the reduction in the profit of the firm that becomes a follower) is insignificant when $m \to N$. Therefore, the indirect effect on the other firms (the second term) alone determines the change in welfare. In this case, this effect is business augmenting: introducing the small number of followers (a marginal reduction of m) increases the rivals' (leaders') output, implying that the second term is negative. Thus, (7) is unambiguously negative (a marginal reduction of m increases welfare).

Proposition 2 states that introducing a small number of followers improves welfare. Thus, Cournot is never best for welfare. This implies the following result.

[13] A similar fashion has been seen in Mankiw and Whinston (1986). The terms "business stealing" and "business augmenting" are taken from this paper.

$x_L^*(m)$	$a(2k+1)/\beta$
$x_F^*(m)$	$a(4k^2+2k(2+n)+1)/\alpha\beta$
$X^*(m)$	$a(4k^2(m+n)+2k(2+n)(m+n)+(mn+m+n))/\alpha\beta$
$\pi_L^*(m)$	$a^2(2k+1)^2(2k^2+k(3+n)+1)/\alpha\beta^2$
$\pi_F^*(m)$	$a^2(k+1)(4k^2+2k(2+n)+1)^2/\alpha^2\beta^2$

Table 1: Results under the linear demand and quadratic cost functions: $\alpha = (2k+n+1)$, $\beta = (4k^2+2k(2+m+n)+(1+m))$ and $n = N-m$.

Corollary of Proposition 2 *Suppose that Assumptions 1–3 are satisfied. Then,* $0, N \notin \operatorname{argmax}_{m\in[0,N]} W^*(m)$.

In other words, the market with both leaders and followers is desirable.[14] Note that HHI is minimized when $m=0$ and $m=N$ since there is no asymmetry in these cases. Thus, this corollary implies that beneficial concentration can generally occur. We further discuss the property of the optimal number of leaders in Section 4.2 by specifying the demand and cost functions.

4 Linear demand and quadratic cost

In this section, we specify the demand and cost functions and further elaborate on the relationship between welfare and the number of leaders. Suppose that the inverse demand function is linear (i.e., $P(X) = a - X$) and the cost function is quadratic (i.e., $C(x) = kx^2$, where $k \geq 0$). Note that if $k = 0$ ($k > 0$), we have a constant (increasing) marginal cost. The results obtained solving the model under these specifications are summarized in Table 1, where $n \equiv N - m$. We can compute the equilibrium social welfare in this specific case by substituting the results of Table 1 into

$$W^*(m) = \frac{1}{2}\left(X^*(m)\right)^2 + m\pi_L^*(m) + n\pi_F^*(m). \tag{8}$$

4.1 The total number of leaders and followers

We demonstrate that leadership reduces welfare under increasing marginal costs if the total number of firms is sufficiently large.

[14]This result contrasts with the result in free entry markets. When the number of followers is endogenously determined, the optimal number of leaders is such that no follower is allowed to enter. See Ino and Matsumura (2012).

Proposition 3 *Assume linear demand and quadratic cost. If $k > 0$, there exists $N' > 0$ such that $\frac{\partial W^*}{\partial m}\big|_{m=0} \lesseqgtr 0$ if and only if $N \gtreqless N'$.*

This proposition implies that when the number of leaders is small and when the number of entire firms is large (i.e., when the ratio of leaders in the industry is small), leadership is more likely to be harmful.

4.2 Optimal number of leaders

We discuss the properties of the optimal number of leaders that maximizes welfare. Let the optimal number of the leaders be m^*, which is given by

$$m^* \in \underset{m \in [0,N]}{\operatorname{argmax}} W^*(m). \tag{9}$$

Proposition 4 *Assume linear demand and quadratic cost. Then, there uniquely exist $m^* \in (0, N)$, and $m^* = N/2$ if $k = 0$ and $m^* > N/2$ if $k > 0$.*

We guess that from the asymmetric result in Proposition 2, welfare can be decreasing in m when m is small, and then welfare turns to be increasing in m, and finally again turns to be decreasing in m (typically, a rotated S-shaped relationship as in Figure 1). Thus, it is plausible that welfare tends to be maximized when m is closer to point $m = N$ than to point $m = 0$. Proposition 4 states that it is true under linear demand and quadratic cost: the optimal number of leaders is larger than $N/2$ (biased towards N).

4.3 Integer constraint

In this subsection, we consider the integer constraint for the number of firms. We show that if the model has linear demand and quadratic cost, we obtain results similar to those of Propositions 2 and 3 even under this integer constraint. Proposition 5(i) indicates that the introduction of one leader into the Cournot model may reduce welfare, as in Proposition 2(i), specifically when the total number of firms in the industry is large as in Proposition 3. Proposition 5(ii) suggests that the introduction of one follower always improves welfare as in Proposition 2(ii).

Proposition 5 *Assume linear demand and quadratic cost. Suppose that $N \in \{2, 3, \cdots\}$. (i) If $k > 0$, then there exists N' (> 2) such that $W^*(1) - W^*(0) \lesseqgtr 0$ if and only if $N \gtreqless N'$. (ii) $W^*(N-1) - W^*(N) > 0$ for all N.*

4.4 Numerical Example

Finally, we present a numerical example. Suppose that $a = 100$, $k = 0.2$, and $N = 10$. Using (8), we simulate how the ratio of leaders in the industry affects social welfare. Table 2 depicts a change in welfare when m changes in $[0, 10]$ given $N = m + n = 10$. Compared to the Cournot case, as predicted in Proposition 3, the welfare is smaller when the ratio of leaders is small, while as predicted in Proposition 2(ii), the welfare is larger when the ratio of leaders is large. From Table 2, we find that the welfare is maximum at $m = 8$.

m	1	2	3	4	5	6	7	8	9
$W^*(m)/W^*(0) - 1$	-0.33%	-0.36%	-0.26%	-0.11%	0.06%	0.23%	0.36%	0.46%	0.43%

Table 2: Welfare increment by leadership

5 Concluding Remarks

Considering multi-leader and multi-follower Stackelberg models, we have shown that the introduction of a small number of leaders into the Cournot model can reduce welfare but that the introduction of a small number of followers always improves welfare.

In this paper, we assume that all firms have an identical cost function and only the difference in their roles yields the asymmetry among firms. If we introduce cost differences among firms, the analysis becomes much more complicated. Leadership by a firm with a lower (higher) marginal cost is more likely to improve (deteriorate) welfare. However, it is not easy to find a more meaningful condition for welfare improvement. Extending our analysis in this direction remains for future research.[15]

[15] For the limit result when an infinite number of followers enter the market under the circumstance where a single leader has a cost advantage, see Ino and Kawamori (2009). Mukherjee and Zhao (2009), Ishida et al. (2011), Pal and Sarkar (2001) and Wang and Mukherjee (2012) also demonstrate that cost heterogeneity plays an important role in

APPENDIX

Proof of Lemma 1:

An increase (decrease) in x_L^* decreases (increases) x_F^* and increases (decreases) X^* given m and n (see the discussion just after equation (4)). Thus, Lemma 1(i) implies Lemma 1(ii) and Lemma 1(iii). As such, we only have to prove Lemma 1(i).

We assume that $x_L^*(m) \leq x_C^*$ and derive a contradiction. Note that $x_L^*(m) \leq x_C^*$ implies that $X^*(m) \leq X_C^*$. Substituting $m = 0$ into (2) yields

$$P'(X_C^*)x_C^* + P(X_C^*) - C'(x_C^*) = 0, \tag{10}$$

where $X_C^* = X^*(0) = Nx_C^* = Nx_F^*(0)$. We compare the LHS of (10) and (1). Assumption 3 ensures that $P'(X)x + P(X)$ is non-increasing in X. Because $X^*(m) \leq X_C^*$, we have $P'(X_C^*)x_C^* + P(X_C^*) - C'(x_C^*) \leq P'(X^*(m))x_C^* + P(X^*(m)) - C'(x_C^*)$. Because $P' - C'' < 0$ and $x_L^*(m) \leq x_C^*$, we have $P'(X^*(m))x_C^* + P(X^*(m)) - C'(x_C^*) \leq P'(X^*(m))x_L^*(m) + P(X^*(m)) - C'(x_L^*(m))$. As is shown in Section 3 (just after equation (5)), $-1 < (N-m)R' < 0$ for $m \in [0, N)$. Thus, we have $P'(X^*(m))x_L^*(m) + P(X^*(m)) - C'(x_L^*(m)) < (1 + (N-m)R')P'(X^*(m))x_L^*(m) + P(X^*(m)) - C'(x_L^*(m))$. Therefore, if (10) is satisfied, the LHS of (1) must be positive, which is a contradiction. **Q.E.D.**

Proof of Proposition 1:

We regard the LHS of (1) as a function $F(m, x_L^*(m), x_F^*(m))$ and the LHS of (2) as a function $G(m, x_L^*(m), x_F^*(m))$. The continuity of $F(\cdot)$ yields

$$\lim_{m\to 0} F(m, x_L^*(m), x_F^*(m)) = F\left(0, \lim_{m\to 0}(x_L^*(m), x_F^*(m))\right).$$

On the other hand, because $F(m, x_L^*(m), x_F^*(m)) = 0\ \forall m \in (0, N)$, $\lim_{m\to 0} F(m, x_L^*(m), x_F^*(m)) = 0$. Hence, $F(0, \lim_{m\to 0}(x_L^*(m), x_F^*(m))) = 0$. Similarly, $G(0, \lim_{m\to 0}(x_L^*(m), x_F^*(m))) = 0$. Therefore, $\lim_{m\to 0}(x_L^*(m), x_F^*(m))$ is a solution of the system of equations (1) and (2) when $m = 0$. Thus, from the

Stackelberg oligopoly.

definitions of $x_F^*(0)$, $x_L^*(0)$, and x_C^*, we have $\lim_{m\to 0} x_F^*(m) = x_F^*(0) = x_C^*$ and $\lim_{m\to 0} x_L^*(m) = x_L^*(0)$. Similarly, taking $m \to N$, we obtain $\lim_{m\to N} x_L^*(m) = x_L^*(N) = x_C^*$ and $\lim_{m\to N} x_F^*(m) = x_F^*(N)$.

Take $m = 0$. Note that the proof of Lemma 1 is also applicable when $m = 0$. Therefore, $x_L^*(0) > x_C^*$. Next, take $m = N$. Note that when $m = N$, nR' vanish from (1). Thus, $x_L^*(N) = x_C^*$ arises only from (10). Then, given $x_L^*(N) = x_C^*$, $x_F^*(N)$ is determined according to (2), i.e., $P'(Nx_C^*)x_F^* + P(Nx_C^*) = C'(x_F^*)$. From (10), $x_F^*(N)$ satisfies this equation if $x_F^*(N) = x_C^*$. Then, the uniqueness of $x_F^*(N)$ implies that $x_F^*(N) = x_C^* = x_L^*(N)$. **Q.E.D.**

Proof of Proposition 2:

(i) The example wherein $P(X) = 1 - X$, $C(x) = x^2$ yields (for more detailed calculation to induce the following equation, see the proof of Proposition 3 because this example is a special case of Proposition 3)

$$\frac{\partial W^*}{\partial m}\Big|_{m=0} = \frac{N(27 + 3N - N^2)}{(3+N)^3(9+2N)^2},$$

which is positive when $N \leq 6$ and negative when $N \geq 7$.

(ii) We show that for any situation satisfying Assumptions 1–3, $\partial W^*(N)/\partial m < 0$. Differentiating (5) and evaluating at $m = N$, we obtain

$$\frac{\partial W^*}{\partial m}\Big|_{m=N} = [Px_L^*(N) - C(x_L^*(N))] - [Px_F^*(N) - C(x_F^*(N))] + N[P - C'(x_L^*(N))]\frac{\partial x_L^*}{\partial m}\Big|_{m=N}.$$

Because $x_L^*(N) = x_F^*(N) = x_C^*$ from Proposition 1(i), the first bracket in the RHS is canceled out. Therefore, if the last term in the RHS is negative, $\partial W^*(N)/\partial m < 0$. The comparative statics using (1) and (2) yields

$$\frac{\partial x_L^*}{\partial m}\Big|_{m=N} = \frac{P'x_C^*(P' - C''(x_C^*))R'|_{m=N}}{N(P' + P''x_C^*)(P' - C''(x_C^*)) + (P' - C''(x_C^*))^2} < 0, \tag{11}$$

where $R'|_{m=N} = -(P'(X_C^*) + P''(X_C^*)x_C^*)/(P'(X_C^*) - C''(x_C^*)) < 0$. We use $x_L^*(N) = x_F^*(N) = x_C^*$ to induce (11). (11) implies that the last term in the RHS is negative. **Q.E.D.**

Proof of Proposition 3:

Differentiating (5) and evaluating at $m = 0$, we obtain (6). From the results in table 1, we have

$$\pi_L^*(0) = P(X^*(0))x_L^*(0) - C(x_L^*(0)) = \frac{(a-c)^2(2k+1)^2(2k^2+k(3+N)+1)}{(2k+N+1)(4k^2+2k(2+N)+1)^2}, \tag{12}$$

$$\pi_F^*(0) = P(X^*(0))x_F^*(0) - C(x_F^*(0)) = \frac{(a-c)^2(k+1)}{(2k+N+1)^2}. \tag{13}$$

From the first-order condition (2),

$$P(X^*(0)) - C'(x_F^*(0)) = -P'(X^*(0))x_F^*(0) = bx_F^*(0) = \frac{a-c}{2k+N+1}. \tag{14}$$

Differentiating x_F^* yields

$$\left.\frac{\partial x_F^*}{\partial m}\right|_{m=0} = \frac{-N(a-c)}{(2k+N+1)^2(4k^2+2k(2+N)+1)}. \tag{15}$$

Substituting (12)–(15) into Equation (6), we obtain

$$\left.\frac{\partial W^*}{\partial m}\right|_{m=0} = \frac{N(a-c)^2 A}{(2k+N+1)^3(4k^2+2k(2+N)+1)^2}, \tag{16}$$

where A is given by

$$A = [8k^3 + 12k^2 + 6k + 1 + (2k^2 + k)N] - kN^2.$$

Because the denominator is positive, the RHS of (16) is negative (positive) if and only if $A < 0$ ($A > 0$). Because $k > 0$, the first term of A, which is within the bracket, is positive, and the second term of A is nonpositive. Note that when $N = 0$, $A > 0$ because the second term is zero. The first term is linear and the second term is quadratic with respect to N. Hence, in absolute value, the second term must exceed the first term when N is greater than some positive number and *vice versa*. **Q.E.D.**

Proof of Proposition 4

First, note that m^* exists in $(0, N)$ because $W^*(m)$ is continuous in m, $[0, N]$ is compact, and $\partial W^*(N)/\partial m < 0$ by Proposition 2(ii).

Substituting the results of Table 1 into and (8) differentiating it yield the following first-order condition for $\max W^*(m)$:

$$\frac{\partial W^*(m)}{\partial m} = \frac{f(m)}{\alpha^3\beta^3} = \frac{k(pm^4 + qm^3 + rm^2 - sm + t) - 2m + N}{\alpha^3\beta^3} = 0, \tag{17}$$

where $\alpha > 0$ and $\beta > 0$ are those defined in Table 1 and

$$p = -a^2(4k+1) < 0,$$
$$q = 12k^2 + 14(N+1)k + 3N + 4,$$
$$r = 3(8k^3 - 4(2N-3)k^2 - 2(3N^2 + 5N - 3)k - N^2 - 3N + 1),$$
$$s = 64k^4 + 16(3N+10)k^3 - 4(3N^2 - 18N - 40)k^2 - 2(5N^3 + 9N^2 - 18N - 40)k - (N^3 + 6N^2 - 6N - 20),$$
$$t = N(32k^4 + 8(3N+10)k^3 + 4(9N+20)k^2 - 2(N^3 + N^2 - 9N - 20)k + (-N^2 + 3N + 10)).$$

Because $\partial^2 W^*(m)/\partial m^2 = f'(m)/\alpha^3\beta^3$ when $f(m) = 0$, both the first-order and the second-order conditions are met if and only if $f(m) = 0$ and $f'(m) < 0$.

When $k = 0$, $f(m)$ is reduced to $-2m + N$, which is zero only if $m = N/2$. Thus, the existence of m^* in $(0, N)$ implies that $m^* = N/2$.

Suppose that $k > 0$. Then, $f(N) < 0$ by Proposition 2(ii) and

$$f\left(\frac{N}{2}\right) = \frac{a^2kN^2}{16}(96k^3 + 24k^2(6+N) + 4k(18+5N) + N^2 + 4N + 12) > 0.$$

Therefore, we have at least one value of m in $(N/2, N)$ that satisfies both the first-order and the second-order conditions; that is, $\exists m' \in (N/2, N)$, $f(m') = 0$ and $f'(m') < 0$. From the shape of $f(m)$, as is shown in the following paragraph, the values of m that satisfy both the first-order and the second-order conditions are never multiple in $(0, N)$. Thus, the existence of m^* in $(0, N)$ implies that $m^* = m' > N/2$.

Because $f(m)$ is quadruplicate with respect to m (typically an inverse W-shaped graph), the number of the values of m satisfying both the first-order and the second-order conditions is at most two (see figure 2). We have already shown that one solution lies on $(N/2, N)$. We then show that the other lies on (N, ∞) if it exists. We solve the equation $f''(m) = 0$, and we obtain

$$m = \frac{12k^2 + 14k(N+1) + 3N + 4 \pm (2k+1)\sqrt{100k^2 + 4k(5N+24) + N^2 + 24}}{16k+4}.$$

We denote the larger solution by $\bar{m}$. Then, we obtain $\bar{m} > N$ by

$$\bar{m} - N = \frac{12k^2 + 14k + 4 + (2k+1)\left(\sqrt{100k^2 + 4k(5N+24) + N^2 + 24} - \sqrt{N^2}\right)}{16k+4} > 0.$$

Therefore, the other value of m satisfying both the first-order and the second-order conditions lies on (N, ∞). **Q.E.D.**

Proof of Proposition 5

(i) The results in table 1 and (8) yield the following equation:

$$W^S(1) - W^S(0) = \frac{a^2(N-1)A}{8(2k+N)(2k+N+1)^2(2k^2+k(2+N)+1)^2}, \tag{18}$$

where A is given by

$$A = \left[\frac{(2k+1)^2}{2k+N} + (2k+1)^2(3+4k) + (4k^2+4k)N\right] - 2kN^2.$$

Because the denominator is positive and because $N \geq 2$, the RHS of (18) is strictly negative if and only if $A < 0$. Because $k > 0$, the first term of A, which is in the bracket, is positive and the second term of A is negative. Note that $A > 0$ when $N = 2$. $(2k+1)^2/(2k+N)$ converges to zero if $N \to \infty$ and the remainder of the elements in the bracket are linear with respect to N. The second term is quadratic with respect to N. Hence, in absolute value, the second term must exceed the first bracket when N is greater than some number that is greater than 2, and *vice versa*.

(ii) The results in table 1 and (8) yield the following equation:

$$W^S(N-1) - W^S(N) = \frac{a^2(N-1)B}{8(k+1)^2(2k+N+1)^2(4k^2+2k(2+N)+N)^2},$$

where B is given by $B = (32k^4 + 72k^3 + 52k^2 + 14k + 1) + (16k^3 + 28k^2 + 14k + 3)N > 0$. **Q.E.D.**

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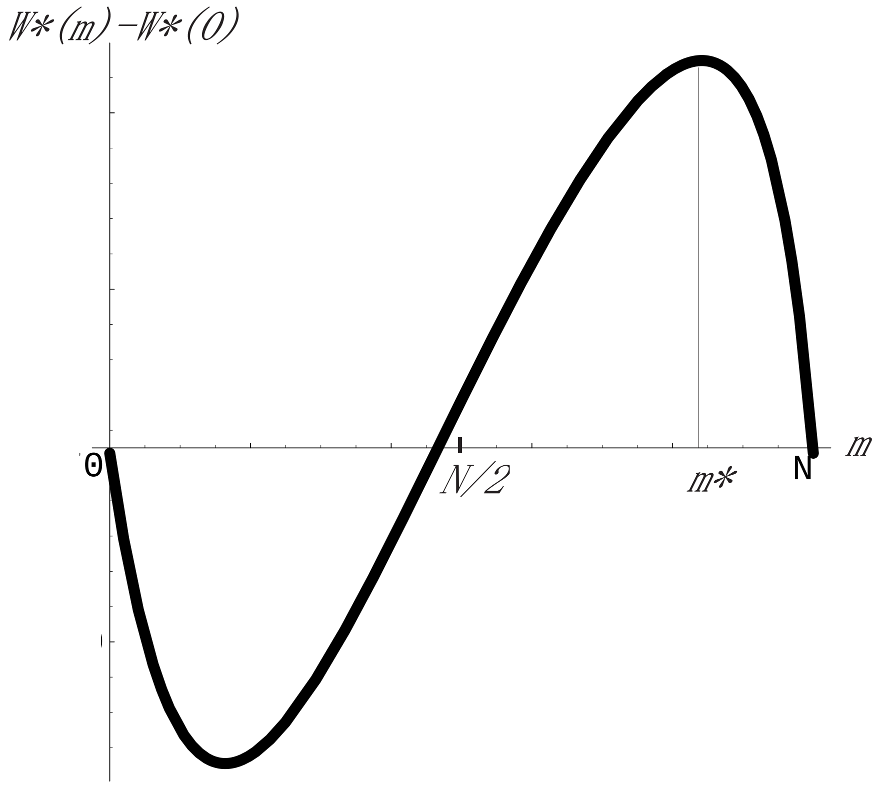

Figure 1: Welfare Comparison when the number of leaders changes for fixed $N = m + n$. When $m = m^*$, welfare is maximized.

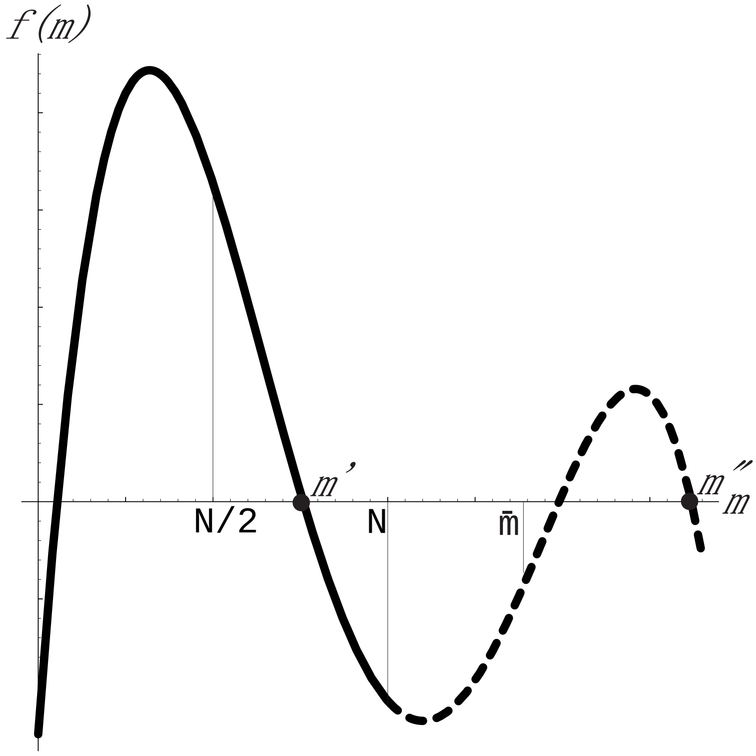

Figure 2: Graph of $f(m)$. When $m = m'$ and $m = m''$, the first-order condition $f(m) = 0$ and the second-order condition $f'(m) < 0$ are simultaneously satisfied.