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A Tentative Consideration of the Three-Period Overlapping Generations Model

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Abstract

This paper considers an overlapping generations model with three periods using the AK model. In overlapping generations models, two-period models are generally more common. While two-period models are easier to analyze, three-period models are more complex and are usually examined through simulations. Needless to say, there are previous studies that have analytically analyzed three-period models. This study develops the methods of previous research by defining the ratio of variables as new variables, thereby reducing the number of variables and deriving the dynamic equations. Furthermore, it explicitly examines the dynamics of the birth rate when both the young and middle-aged generations engage in fertility behavior, offering a different perspective from the conventional assumption that only one generation engages in fertility behavior. Although this model setting can also be found in previous studies, this paper demonstrates that childcare support policies targeting a specific generation may reduce the birth rate of other generations. Additionally, it derives the equilibrium solution of a three-period overlapping generations model that incorporates endogenous fertility and capital accumulation.

Keywords: Endogenous Fertility, Overlapping Generations Model

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† Any remaining errors are solely the responsibility of the author.

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1. Introduction

Overlapping generations (OLG) models are typically formulated with two periods. Although three-period models also exist, in many of these, individuals do not work or participate in capital accumulation during the first period, nor do they make economic decisions—yet they are still alive. As such, these models are sometimes described as three-period models. In contrast, the two-period model, which divides life into a working (young) period and a retirement (old) period, is more straightforward in terms of economic interpretation and is widely used in OLG modeling. Moreover, in the case of the two-period model, solving the dynamic equations is relatively simple. While multi-period OLG models also exist, analytical investigation becomes more difficult, and such models are often examined through simulation-based analysis.

This paper demonstrates that a three-period overlapping generations (OLG) model can be analyzed by incorporating an AK-type production function. A three-period OLG model is also considered in Tanaka (2013), where the number of variables is reduced by redefining certain ratios as new variables, thereby enabling the solution of the dynamic equations. As an application of the three-period OLG model, this paper derives the dynamic equation for fertility when both the younger and middle-aged generations engage in fertility behavior. Most existing models assume that only a single generation is responsible for fertility decisions. However, a model in which multiple generations make fertility decisions is also found in d'Albis, Greulich, and Ponthiere (2018). This paper explicitly models the dynamics of fertility and further shows that childcare support policies targeting a specific generation can reduce the fertility rate of other generations.

Blanchard and Fischer (1989) show the perpetual youth model as the other kind of overlapping generations model. The perpetual youth model assumes a constant mortality rate and a constant influx of new individuals into the population. As a result, at any given point in time, people are born at different times, leading to the coexistence of individuals from various cohorts. In this sense, the model can be regarded as an overlapping generations model with multiple cohorts. However, since this model does not explicitly distinguish between generations, it is difficult to conduct policy analysis targeting a specific generation. Therefore, extending the standard two-period overlapping generations model to a three-period version enables policy analysis focused on a particular generation, allowing for more targeted and detailed examination of generational policies. Thus, constructing such a model is considered highly important.

The structure of this paper is as follows. Section 2 sets up a three-period overlapping generations model that incorporates capital accumulation, and Section 3 derives the equilibrium solution. Section 4 introduces a three-period OLG model with endogenous fertility in a small open economy and derives its equilibrium. Section 5 examines childcare support policies targeted at specific generations and demonstrates that such policies do not necessarily lead to an increase in fertility. Section 6 derives the equilibrium solution for a three-period OLG model that considers both capital accumulation and endogenous fertility. Section 7 concludes the paper.

2. Model

2.1 Households

Consider an individual who lives through three periods: youth, middle age, and old age. In each period, three overlapping generations - young, middle-aged, and old - coexist. We consider a three-period overlapping generations (OLG) model. Here, we normalize the population size of each generation for one and assume there is no population growth.

The utility function of an individual in a household is assumed to be a logarithmic utility function as follows:

$$u_t = \alpha ln c_{1t} + \beta ln c_{2t+1} + (1-\alpha-\beta) ln c_{3t+2}, 0<\alpha<1, 0<\beta<1, \alpha+\beta<1. \quad (1)$$

Here, c_{1t} denotes the consumption of the young generation, c_{2t+1} the consumption of the middle-aged generation, and c_{3t+2} the consumption of the old generation. An individual who belongs to the young generation in period t becomes middle-aged in period $t+1$, and old in period $t+2$.

There are three budget constraint equations corresponding to the youth, middle age, and old age periods. The budget constraint in the youth period is given as follows:

$$(1-\tau)w_t = c_{1t} + s_{1t} \quad (2)$$

s_{1t} denotes savings, τ is the pension contribution rate, and w_t is the wage rate.

The budget constraint in the middle age period is as follows:

$$(1+r_{t+1})s_{1t} + (1-\tau)w_{t+1} = c_{2t+1} + s_{2t+1} \quad (3)$$

r_{t+1} denotes the interest rate.

The budget constraint in the old age period is as follows:

$$(1+r_{t+2})s_{2t+1} + P_{t+2} = c_{3t+2} \quad (4)$$

P_{t+1} denotes the pension benefit.

Because of (2)-(4), the lifetime budget constraint is shown as follows.

$$(1-\tau)w_t + \frac{(1-\tau)w_{t+1}}{1+r_{t+1}} + \frac{P_{t+2}}{(1+r_{t+1})(1+r_{t+2})} = c_{1t} + \frac{c_{2t+1}}{1+r_{t+1}} + \frac{c_{3t+2}}{(1+r_{t+1})(1+r_{t+2})} \quad (5)$$

The diagram below illustrates the generational structure of the model.

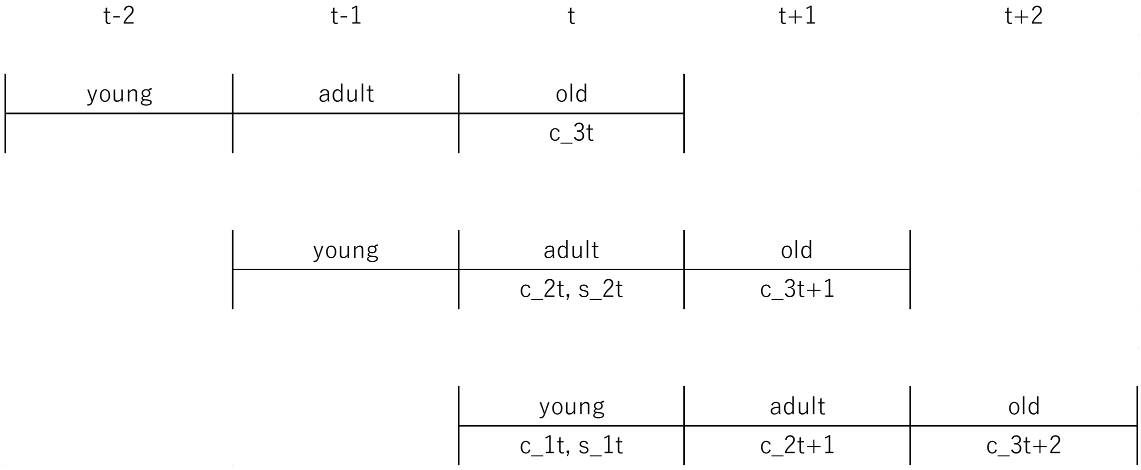

Figure 1: Generation Structure

By maximizing the utility function (1) subject to the lifetime budget constraint (5), the optimal consumption allocation is as follows:

$$c_{1t} = \alpha\left((1-\tau)w_t + \frac{(1-\tau)w_{t+1}}{1+r_{t+1}} + \frac{P_{t+2}}{(1+r_{t+1})(1+r_{t+2})}\right) \tag{6}$$

$$c_{2t+1} = (1+r_{t+1})\beta\left((1-\tau)w_t + \frac{(1-\tau)w_{t+1}}{1+r_{t+1}} + \frac{P_{t+2}}{(1+r_{t+1})(1+r_{t+2})}\right) \tag{7}$$

$$c_{3t+2} = (1+r_{t+1})(1+r_{t+2})(1-\alpha-\beta)\left((1-\tau)w_t + \frac{(1-\tau)w_{t+1}}{1+r_{t+1}} + \frac{P_{t+2}}{(1+r_{t+1})(1+r_{t+2})}\right) \tag{8}$$

2.2 Firms

Firms produce final goods using capital stock and labor. The production function is assumed to be a Cobb-Douglas type, as shown below:

$$Y_t = AK_t^{\theta}(B_tL_t)^{1-\theta} \tag{9}$$

K_t and L_t denotes capital stock and labor, respectively. It is assumed $B_t = \frac{K_t}{L_t}$. In addition, following Grossman and Yanagawa (1993), we assume the existence of capital accumulation externalities, where the per capita capital stock affects productivity. This type of setting can be regarded as similar to models that incorporate capital externalities, such as Romer (1990).

Under the assumption of a perfectly competitive market, the wage rate and the interest rate are determined as follows, assuming a normalized population size of one. It is also assumed that the capital stock fully depreciates within one period.

$$w_t = (1-\theta)AK_t \tag{10}$$

$$1 + r_t = \theta A \tag{11}$$

This model takes the form of an AK model, in which the marginal productivity of capital does not diminish, and the wage rate is a linear function of the capital stock. In this way, by incorporating capital externalities, the model can be represented as an AK model.

2.3 Government

The government operates the pension system on a pay-as-you-go basis and provides benefits. These benefits are distributed to the elderly generation, while the burden of funding them is borne by the young and middle-aged generations.

$$N_{t-1}P_{t+1} = \tau(N_t w_{t+1} + N_{t+1} w_{t+1}) \rightarrow P_{t+1} = \tau(n + n^2) w_{t+1} \tag{12}$$

N_{t-1}, N_{t+1} and N_t the number of elderly individuals in period $t+1$, the number of young individuals in period $t+1$ and the number of middle-aged individuals in period t+1. n denotes the population ratio between successive generations. However, since this paper normalizes the population size to one and assumes no population growth, it follows that $P_{t+1} = 2\tau w_{t+1}$.

3. Equilibrium

The equilibrium solution of the model economy can be derived from the initial capital stock K_0 and the capital accumulation equation. First, the capital accumulation equation can be expressed as follows:

$$K_{t+1} = s_{1t} + s_{2t} \tag{13}$$

It is important to note that it is the sum of the first-period savings by the young generation, s_{1t}, and the second-period savings by the middle-aged generation, s_{2t}. These can be expressed as follows:

$$s_{1t} = (1-\tau)w_t - \alpha\left((1-\tau)w_t + \frac{(1-\tau)w_{t+1}}{1+r} + \frac{P_{t+2}}{(1+r)^2}\right) \tag{14}$$

$$s_{2t} = (1+r)(1-\alpha-\beta)\left((1-\tau)w_{t-1} + \frac{(1-\tau)w_t}{1+r} + \frac{P_{t+1}}{(1+r)^2}\right) - \frac{P_{t+1}}{1+r} \tag{15}$$

By substituting (14) and (15) into (13), the following dynamic equation of capital can be derived.

$$\begin{aligned} K_{t+1} = {} & (2(1-\alpha)-\beta)(1-\tau)(1-\theta)AK_t - (\alpha(1-\tau) + 2(\alpha+\beta)\tau)\frac{(1-\theta)A}{1+r}K_{t+1} \\ & + (1+r)(1-\alpha-\beta)(1-\tau)(1-\theta)AK_{t-1} - \frac{2\alpha\tau(1-\theta)A}{(1+r)^2}K_{t+2} \end{aligned} \tag{16}$$

Here, the variables are defined as follows:

$$\bar{K}_1 = 1 - (\alpha(1-\tau) + 2(\alpha+\beta)\tau)\frac{(1-\theta)A}{1+r} \tag{17}$$

$$\bar{K}_2 = \frac{2\alpha\tau(1-\theta)A}{(1+r)^2} \tag{18}$$

$$\bar{K}_3 = (2(1-\alpha)-\beta)(1-\tau)(1-\theta)A \tag{19}$$

$$\bar{K}_4 = (1+r)(1-\alpha-\beta)(1-\tau)(1-\theta)A \tag{20}$$

Here, letting $\frac{K_{t+1}}{K_t} = g_t$, the growth rate equation can be derived as follows:

$$\bar{K}_1 g_t + \bar{K}_2 g_t g_{t+1} = \bar{K}_3 + \frac{\bar{K}_4}{g_{t-1}} \tag{21}$$

The dynamic equation, which includes endogenous variables over four periods, can be expressed as a three-period dynamic equation by using ratio variables.

3.1 Case of $\boldsymbol{\tau = 0}$

First, consider an economic model without a pension system. In this case, $\bar{K}_2 = 0$ in equation (21), which implies that the dynamic equation becomes a two-period equation involving g_t and g_{t-1}. At the steady state, the growth rate g can be obtained by substituting $g_t = g_{t+1} = g$ into equation (21) and solving the resulting quadratic equation, as shown below:

$$g = \frac{\bar{K}_3 + \sqrt{\bar{K}_3^{\,2} + 4\bar{K}_1\bar{K}_4}}{2\bar{K}_1} \tag{22}$$

Then, the condition for local stability is $-1 < \frac{dg_t}{dg_{t-1}} < 1$. Given that $\frac{1}{\bar{K}_3 g} > 0$ and $\frac{dg_t}{dg_{t-1}} = \frac{1}{\bar{K}_3 g}$, the specific condition can be expressed as follows:

$$-1 < \frac{1}{\bar{K}_3 g} < 1 \tag{23}$$

Furthermore, if the steady-state growth rate g is less than 1, the capital stock will continue to decline over time, resulting in a contracting equilibrium. To rule out such an equilibrium, the condition $1 < g$ is required.

It has been shown that even in the case of the three-period overlapping generations model, the equilibrium solution and stability can be relatively easily derived by considering a model linearized by the AK model and presenting the dynamic equation of the growth rate.

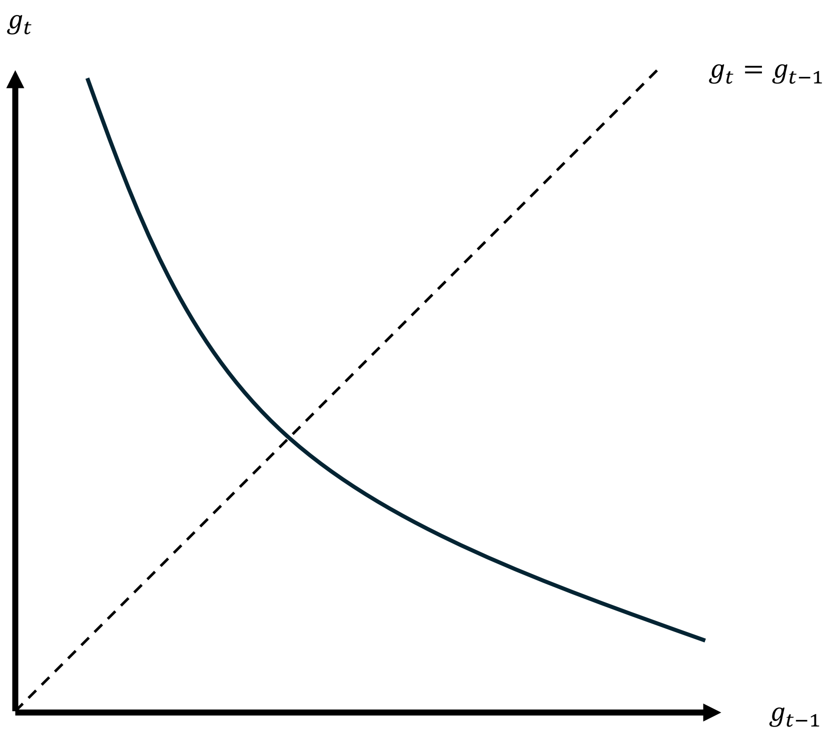

Figure 2: Dynamics of g_t

3.2 Case of $\tau \neq 0$

We now consider the steady-state growth rate when a small pension contribution rate is introduced, starting from the case without a pension system. By totally differentiating equation (20) with respect to g and τ in the approximation of $\tau = 0$, we obtain the following equation:

$$\frac{dg}{d\tau} = -\frac{\frac{2\alpha}{1+r}g^3 + \frac{(\alpha+2\beta)}{1+r}g^2 + (2(1-\alpha)-\beta)g - (1+r)(1-\alpha-\beta)}{\frac{1}{(1-\theta)A} + \frac{\alpha}{1+r} - (2(1-\alpha)-\beta)} \tag{24}$$

The sign of equation (24) is indeterminate. However, it is generally known that the introduction of a pay-as-you-go pension system tends to reduce capital accumulation, suggesting that $\frac{dg}{d\tau} < 0$. When θ is small—in other words, when the capital share is low—it can be shown that $\frac{dg}{d\tau} < 0$ holds.

From this, we obtain the following Proposition:

Proposition 1

The equilibrium solution of the three-period overlapping generations model exists. Furthermore, an increase in the pension contribution rate lowers the economic growth rate through a decline in capital accumulation.

In general, since a pay-as-you-go pension system tends to reduce capital accumulation, it is strongly suggested that $\frac{dg}{d\tau} < 0$. However, the sign is not definitively determined.

4. Application to the Dynamics of the Fertility

There exists a substantial body of prior research that considers endogenous fertility within overlapping generations (OLG) models, beginning with studies such as van Groezen, Leers, and Meijdam (2003). However, these models typically assume that fertility decisions are made only during one period of an individual's life. By relaxing this assumption—that is, by allowing fertility decisions to be made over two periods—a new economic model can be constructed in which the dynamic equation of the birth rate can be derived.

In this section, we assume a small open economy. As a result, the interest rate is equal to the world interest rate, and capital accumulation no longer needs to be considered. This simplification allows for a more tractable analysis.

Now, we proceed to explain the model. The utility function is assumed to take the following form:

$$u_t = \alpha lnc_{1t} + \beta lnc_{2t+1} + \gamma lnn_{1t} + \delta lnn_{2t+1} + (1-\alpha-\beta-\gamma-\delta)lnc_{3t+2} \tag{25}$$

Where $0 < \alpha < 1, 0 < \beta < 1, 0 < \gamma < 1, 0 < \delta < 1, \alpha+\beta+\gamma+\delta < 1$.

Let n_{1t} denote the number of children born during the young period, and n_{2t+1} denote the number of children born during the middle-aged period.

It is assumed that individuals can have and raise children during both the young and middle-aged periods of their lives. Child-rearing is completed within a single period: if a child is raised during the young period, they enter the young generation when the parent reaches middle age; if a child is raised during the middle-aged period, they become part of the young generation when the parent reaches old age.

We now turn to the budget constraints. The budget constraint during the young period is as follows:

$$w = c_{1t} + z_t n_{1t} + s_{1t} \tag{26}$$

where z_t denotes the child-rearing cost per child during the young period.

The budget constraint during the middle-aged period is as follows:

$$(1+r)s_{1t} + w = c_{2t+1} + z_{t+1}n_{2t+1} + s_{2t+1} \tag{27}$$

The budget constraint during the old-age period is as follows:

$$(1+r)s_{2t+1} = c_{3t+2} \tag{28}$$

From equations (26) to (28), the lifetime budget constraint can be expressed as follows:

$$w + \frac{w}{1+r} = c_{1t} + z_t n_{1t} + \frac{c_{2t+1}}{1+r} + \frac{z_{t+1} n_{2t+1}}{(1+r)} + \frac{c_{3t+2}}{(1+r)^2} \tag{29}$$

By maximizing the utility function (25) subject to the lifetime budget constraint (29), we obtain the following optimal allocation:

$$c_{1t} = \alpha \left(w + \frac{w}{1+r} \right) \tag{30}$$

$$c_{2t+1} = (1+r)\beta \left(w + \frac{w}{1+r} \right) \tag{31}$$

$$c_{3t+2} = (1+r)^2 (1 - \alpha - \beta - \gamma - \delta) \left(w + \frac{w}{1+r} \right) \tag{32}$$

$$n_{1t} = \frac{\gamma}{z_t} \left(w + \frac{w}{1+r} \right) \tag{33}$$

$$n_{2t+1} = \frac{(1+r)\delta}{z_{t+1}} \left(w + \frac{w}{1+r} \right) \tag{34}$$

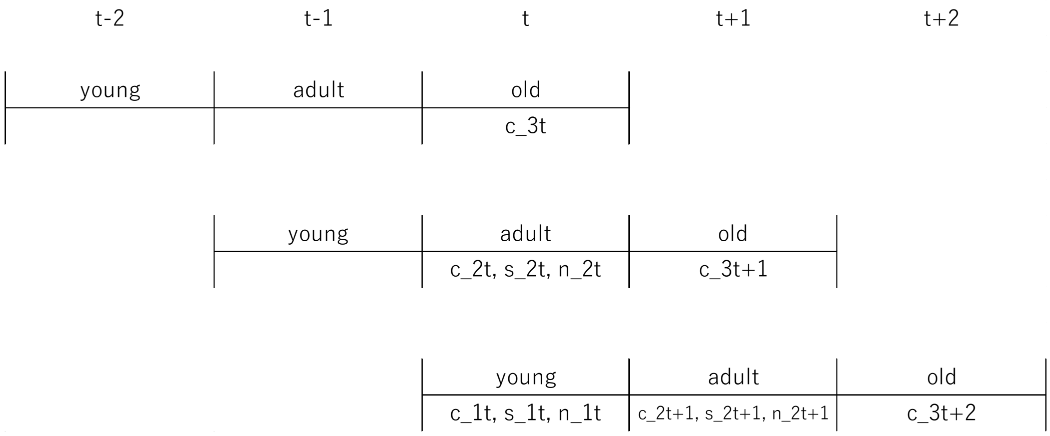

Figure 3: The Structure of Generations

Let N_t be the population size of the young generation in period t. At this time, the population size of the working-age (middle-aged) generation in period t is represented as N_{t-1}. The population size of the young generation in period $t+1$, denoted as N_{t+1}, is determined by the total number of children born to each generation. Furthermore, if we define the intergenerational population ratio as $\frac{N_{t+1}}{N_t} = n_t$, the dynamic equation of the birth rate can be expressed as follows:

$$N_{t+1} = N_t n_{1t} + N_{t-1} n_{2t} \rightarrow n_t = n_{1t} + \frac{n_{2t}}{n_{t-1}} \tag{35}$$

Here, suppose that the childcare costs z_t and z_{t+1} are given as exogenous variables z. In this case, the dynamics of the birth rate can be illustrated as shown in the following diagram.

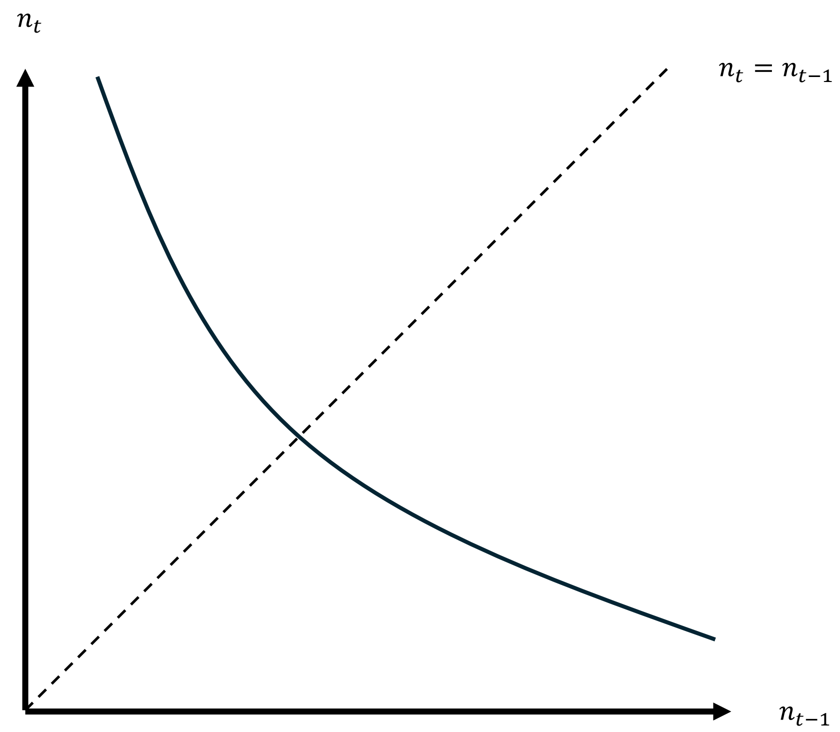

Figure 4: Dynamics of n_t

The fertility at the steady state n is given by $n_t = n_{t-1} = n$. Because of (35), the fertility at the steady state is given by the follows as the other form.

$$n = \frac{n_1 + \sqrt{n_1^2 + 4n_2}}{2} \tag{36}$$

The local stability condition $\frac{dn_t}{dn_{t-1}} = -\frac{n_{2t}}{n^2}$ is given by the follows.

$$-1 < -\frac{n_{2t}}{n^2} < 0 \tag{37}$$

Then, the following Proposition can be established.

Proposition 2

A dynamic equilibrium solution for the birth rate in the three-period overlapping generations model exists. Given the population growth rate at the initial point, the population growth rate for each period can be determined.

5. Analysis Considering Child Allowances

Using the endogenous fertility model in a small open economy presented in Section 4, we analyze the impact of child allowances on the birth rate. Here, we assume that childcare support policies are implemented for the young generation. The childcare support policy considered in this context targets a specific generation. When fertility behavior spans more than one period, it becomes possible to analyze the effects of such targeted childcare support policies.

The financial resources for the childcare support policy are assumed to come from taxation, with the tax burden placed not only on the young generation but also on the working-age (middle-aged) generation. In this case, the lifetime budget constraint is expressed as follows:

$$(1-\tau)w + \frac{(1-\tau)w}{1+r} = c_{1t} + (z_t - q)n_{1t} + \frac{c_{2t+1}}{1+r} + \frac{z_{t+1}n_{2t+1}}{(1+r)} + \frac{c_{3t+2}}{(1+r)^2} \tag{38}$$

Here, let τ represent the tax rate and q the amount of the child allowance. Also, assume that $z_t = z_{t+1} = z$. Under these conditions, the birth rates during the young and middle-aged periods are respectively given as follows:

$$n_{1t} = \frac{\gamma}{z-q}\left((1-\tau)w + \frac{(1-\tau)w}{1+r}\right) \tag{39}$$

$$n_{2t+1} = \frac{(1+r)\delta}{z}\left((1-\tau)w + \frac{(1-\tau)w}{1+r}\right) \tag{40}$$

The government's budget constraint related to child allowances is given as follows. Child allowance benefits are provided under a balanced budget without issuing public debt.

$$N_t n_{1t} q = N_t \tau w + N_{t-1}\tau w \rightarrow n_{1t} q = \left(1 + \frac{1}{n_{t-1}}\right)\tau w \tag{41}$$

We examine whether n_t can be increased given n_{t-1}. We consider an infinitesimal increase in the child allowance from zero. Based on equations (35), (39), (40), and (41), the following expression can be derived.

$$\frac{dn_t}{d\tau} = \frac{w}{z}\left(1 + \frac{1}{n_{t-1}} - \frac{(\gamma + (1+r)\delta)(2+r)}{1+r}\right) \tag{42}$$

Then, the following proposition can be established.

Proposition 3

Childcare support policies targeting a specific generation may lead to a decline in birth rates among other generations, resulting in a decrease in the total fertility rate.

If $\frac{dn_t}{d\tau} = \frac{w}{z}\left(1 + \frac{1}{n_{t-1}} - \frac{(\gamma+(1+r)\delta)(2+r)}{1+r}\right) < 0$, Even if the child allowance increases the birth rate of the younger generation, the negative effect on the birth rate of the older generation may be larger, resulting in an overall decline in the birth rate. When n_{t-1} is small, $\frac{dn_t}{d\tau} < 0$ does not occur, but when n_{t-1} is large, it is possible that $\frac{dn_t}{d\tau} > 0$.

6. Fertility Dynamics in a Closed Economy

Let the childcare costs in the young and middle-aged periods be given by $z_t = \bar{z}w_t$ and $z_{t+1} = \bar{z}w_{t+1}$, respectively. Then, the fertility rates in the young and middle-aged periods are shown as follows, respectively.

$$n_{1t} = \frac{\gamma}{\bar{z}}\left(1 + \frac{g_t}{1 + r_{t+1}}\right) \tag{43}$$

$$n_{2t+1} = \frac{(1 + r_{t+1})\delta}{\bar{z}}\left(\frac{1}{g_t} + \frac{1}{1 + r_{t+1}}\right) \tag{44}$$

We define $g_t = \frac{k_{t+1}}{k_t}$ in the population growth model. Then, the wage rate (10) is given by $w_t = (1-\theta)Ak_t$. Furthermore, the dynamic equation of capital is given as follows.

$$\begin{aligned} n_t k_{t+1} = &\left(1 - \alpha + \frac{1 - \alpha - \beta}{n_{t-1}}\right)(1-\theta)Ak_t - \frac{\alpha(1-\theta)A}{1+r}k_{t+1} \\ &+ \frac{(1+r)(1-\alpha-\beta)(1-\theta)Ak_{t-1}}{n_{t-1}} \end{aligned} \tag{45}$$

The equilibrium solution in this economic model is characterized by the following two dynamic equations.

$$g_t = \frac{\left((1-\alpha) + \frac{1-\alpha-\beta}{n_{t-1}}\right)(1-\theta)A + \frac{(1+r)(1-\alpha-\beta)(1-\theta)A}{n_{t-1}}\frac{1}{g_{t-1}}}{n_t + \frac{\alpha(1-\theta)A}{1+r}} \tag{46}$$

$$n_t = \frac{\gamma}{\bar{z}}\left(1 + \frac{g_t}{1+r}\right) + \frac{(1+r)\delta}{\bar{z}}\left(\frac{1}{g_{t-1}} + \frac{1}{1+r}\right)\frac{1}{n_{t-1}} \tag{47}$$

In the steady state, g and n are given so as to satisfy the following equations, respectively.

$$g = \frac{\left((1-\alpha) + \frac{1-\alpha-\beta}{n}\right)(1-\theta)A + \sqrt{\left(1-\alpha+\frac{1-\alpha-\beta}{n}\right)^2(1-\theta)^2A^2 + 4\left(1 + \frac{\alpha(1-\theta)A}{1+r}\frac{1}{n}\right)(1+r)(1-\alpha-\beta)(1-\theta)A}}{2\left(n + \frac{\alpha(1-\theta)A}{1+r}\right)} \tag{48}$$

$$n = \frac{\frac{\gamma}{\bar{z}}\left(1 + \frac{g}{1+r}\right) + \sqrt{\left(1 + \frac{g}{1+r}\right)^2\frac{\gamma^2}{\bar{z}^2} + 4\left(\frac{1}{g} + \frac{1}{1+r}\right)\frac{(1+r)\delta}{\bar{z}}}}{2} \tag{49}$$

From equation (48), it can be seen that g is a decreasing function of n. On the other hand, in equation (49), the sign of $\frac{dn}{dg}$ is not uniformly determined. Nevertheless, when n is small, it can be considered that a steady state exists in the form illustrated in the figure below.

$$\frac{dn}{dg} = \frac{1}{2}\left(\frac{\gamma}{\bar{z}}\frac{1}{1+r} + \frac{1}{2}\frac{\frac{2\gamma^2}{\bar{z}^2}\frac{1}{1+r}\left(1+\frac{g}{1+r}\right) - \frac{(1+r)\delta}{\bar{z}}\frac{4}{g^2}}{\sqrt{\left(1+\frac{g}{1+r}\right)^2\frac{\gamma^2}{\bar{z}^2} + 4\left(\frac{1}{g}+\frac{1}{1+r}\right)\frac{(1+r)\delta}{\bar{z}}}}\right) \tag{50}$$

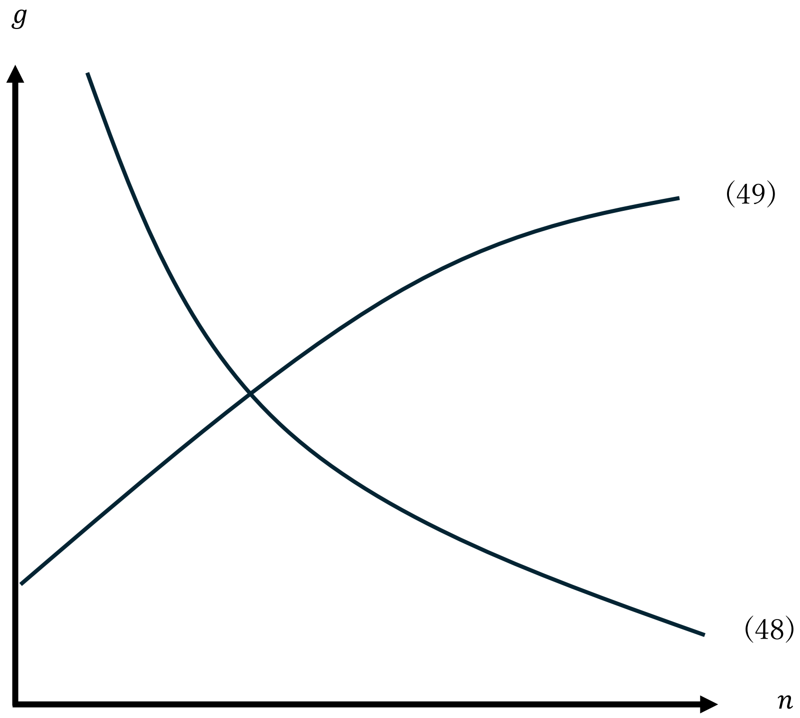

Figure 5: g and n at the steady state

The stability condition is shown as follows.

$$\begin{pmatrix} g_t - g \\ n_t - n \end{pmatrix} = \begin{pmatrix} a' & b' \\ c & d \end{pmatrix} \begin{pmatrix} g_{t-1} - g \\ n_{t-1} - n \end{pmatrix} \tag{51}$$

$$a = -\frac{1}{\frac{\alpha(1-\theta)A}{1+r} + n}\frac{(1+r)(1-\alpha-\beta)(1-\theta)A}{n}\frac{1}{g^2} \tag{52}$$

$$b = (1-\alpha-\beta)(1-\theta)A\left(1+\frac{1+r}{g}\right)\frac{1}{n^2}\frac{1}{\frac{\alpha(1-\theta)A}{1+r} + n} \tag{53}$$

$$c = \frac{\frac{\gamma}{\bar{z}}\frac{a}{1+r} - \frac{(1+r)\delta}{\bar{z}g^2 n}}{1 - e\frac{\gamma}{\bar{z}}\frac{1}{1+r}} \tag{54}$$

$$d = \frac{\frac{\gamma}{\bar{z}}\frac{b}{1+r} - \frac{(1+r)\delta}{\bar{z}n^2}\left(\frac{1}{g} + \frac{1}{1+r}\right)}{1 - e\frac{\gamma}{\bar{z}}\frac{1}{1+r}} \tag{55}$$

$$e = -\frac{g}{n + \frac{\alpha(1-\theta)A}{1+r}} \tag{56}$$

$$a' = a + ce \tag{57}$$

$$b' = b + de \tag{58}$$

If $-2 < a' + d < 2$ and $-1 < a'd - b'c < 1$, the steady state equilibrium is sink[1]

7. Conclusions

Typically, overlapping generations (OLG) models are formulated with two periods. This is because two-period models are often used due to their relative simplicity in solving dynamic equations. In the case of a three-period OLG model, the dynamic equation takes the form of a second-order difference equation. However, by adopting an AK-type model, it is possible to transform the second-order difference equation into a first-order one, allowing for an analytical derivation of the equilibrium solution. As an application of the three-period OLG model, a fertility behavior model involving two generations can be considered. Among the elderly, there are two types: the "young-old" and the "old-old." By constructing a three-period OLG model, it becomes possible to analyze which type of elderly population is being targeted in the implementation of policies.

[1] See Azariadis (1993) for checking the stability condition.

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