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Optimal Nonlinear Income Taxation for Non-Cooperative Couples

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Abstract

This study examines optimal nonlinear income taxation on non-cooperative couples that under-provide a household public good. In this study, the income tax has the role of improving the under-provision of the household public good in addition to equity consideration and revenue collection. The optimal marginal tax rate is characterized by the well-known Mirrleesian *ABC* term and new Pigouvian term. The Pigouvian term can be further decomposed by the two parts. The first reflects the effects of improving the under-provision of the household public good, while the second relates to the expansion of the income tax flexibility. The Pigouvian term results in the marginal tax rate on the top earner being positive. Using US wage data, our quantitative analysis shows that the existence of non-cooperative behavior raises the optimal marginal tax rates at any income level. This result suggests that the optimal marginal tax rates derived in previous studies, which disregard noncooperative behavior, may have been lowly estimated.

Keywords: Optimal Nonlinear Income Taxation, Non-Cooperative Behavior, Household Public Good

Classification: H21, J13, J16

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1 Introduction

The provision of household public goods such as childbirth, early childhood education, childcare, housing maintenance, and health management are extremely important economic activities for people involved in family formation. The value of household production, as an indicator related to household public goods, amounts to between about 40% and 60% of GDP in the major industrialized countries (Ahmad and Koh, 2011). Household public goods also generate positive externalities for family members as well as for other households or even for society (Heckman, 2006; Heckman and Masterov, 2007).[1]

Time and effort invested by household members remain crucial factors in the production of household public goods, even today, despite the availability of external substitutes for housework. For instance, Del Boca et al. (2014) and Lundborg et al. (2014) empirically demonstrate that the time invested by both male and female spouses is crucial for human capital accumulation in children. However, as time and effort cannot be effectively monitored between the spouses, the couple can potentially engage in non-cooperative behavior in housework and childcare (Del Boca and Flinn, 2012; Cochard et al., 2016).[2] The strategic interactions between spouses in providing household public goods lead to under-provision, that is, free-rider problems. Doepke and Kindermann (2019) empirically conclude that non-cooperative behavior between spouses leads to a lower fertility rate, and Ashraf (2009) shows that inefficiencies can arise in spouses' saving decisions using an experimental method. The free-rider problem within households can also occur in housing maintenance, health maintenance related to insurance prices, and informal caregiving.

This study focuses on income taxation as a tool to encourage more housework time to correct the free-rider problems in housework and childcare. Kabátek et al. (2014) provide empirical evidence showing that income taxes can directly manipulate the choice between labor supply to the external market and housework. Specifically, their empirical result shows a substitute relationship between labor supply in the external market and housework and childcare time.

Given the importance of time and effort for household public goods and the role of income taxation to encourage housework time, we analyze the optimal nonlinear income taxation for non-cooperative couples whose behavior leads to the under-provision of household public goods. In this study, income tax serves to improve the under-provision in addition to revenue collection and equity considerations. Although numerous studies have analyzed optimal nonlinear income taxation for couples or families, a common assumption is to employ a unitary or collective model, in which household allocations are efficient. Conversely, the present study considers spouses' non-cooperative behavior that leads to the under-provision. To the best of our knowl-

[1] Heckman (2006) and Heckman and Masterov (2007) empirically show that child quality impacts health conditions in a local area, social skills, and crime rates.

[2] Del Boca and Flinn (2012) empirically show that one-quarter of couples engage in non-cooperative behavior. Using an experimental method, Cochard et al. (2016) demonstrate that having children and being married decreases cooperation.

edge, this study is the first to incorporate non-cooperative household behavior into the Mirrleesian framework for optimal nonlinear income taxation.

This study's model considers households comprising two members. They non-cooperatively contribute to the provision of a household public good, thus inefficiently under-providing. As our model allows for a substitute relationship between the labor supply and household work time, an increase in the income tax rate incentivizes income earners to allocate more time toward housework.

Two points should be noted regarding taxation. First, this study considers an individual tax system, following the fact that most countries, such as England, Canada, and Japan, have adopted individualized income tax system. Although in the U.S. and Germany, couples can choose either individual or joint taxation (couple taxation), the recent trend of positive assortative mating strongly favors individual taxation.[3] Second, this study considers the cases with and without tagging gender to enrich the analysis and derive a clear intuition. The former refers to a gender-based tax system, which has been analyzed in numerous studies (e.g., Boskin and Sheshinski, 1983; Cremer et al., 2010; Alesina et al., 2011; Meier and Rainer, 2015; Obara and Ogawa, 2024).[4] The latter is a realistic and commonly used income tax system in almost all countries, where individuals earning the same amount of income are subject to the same tax rate, whether they are part of a married couple or not, and regardless of gender.

The multidimensional nature of household types poses well-known technical difficulties in a screening context. For instance, Kleven et al., (2009) restricts the labor supply of the second earner to avoid this problem. Recently, Alves et al. (2024) provide specifications to identify household productivity as one screening variable in the model with three dimensions of characteristics (the two spouses' productivities and bargaining weight parameter). However, to do so, they assume a collective decision of the household, which leads to an efficient household allocation. Conversely, to focus on the under-provision of the household public good due to non-cooperative behavior, we adopt a different method to address the issue of multiple dimensions. In this study, we assume assortative mating of the spouses, as in Cremer et al. (2016), so that our model can treat one-dimensional problems while incorporating non-cooperative behavior. The positive assortative mating of couples with respect to income is rapidly progressing in developed countries and causes serious inequality among households (Carbone and Cahn, 2014; Eika et al., 2019).

The study provides the $ABC+D(1+E)$ formula for the optimal marginal tax rates for the non-cooperative couples: the first is the well-known Mirrleesian ABC form,

[3]As positive assortative mating (i.e., the declining income gap between spouses) is rapidly progressing in developed countries, the advantage of choosing couple taxation has become smaller from the perspective of tax avoidance.

[4]According to a Vox column (https://voxeu.org/article/gender-based-taxation-response-critics#fn1), Alesina, Ichino, and Karabarbounis indicated that the gender-based tax system has been widely and intensely discussed in several European countries including Spain, Italy, Germany, Austria, France, and Denmark. The opposition party in Spain proposed gender-based taxation in its campaign platform.

which considers the tax-induced distortions and inequality due to wage distribution, and the second is the new Pigouvian $D(1+E)$ form. The term D captures the effects of improving the under-provision of the household public good, and the term E reflects the extent to which the flexibility of income taxation on a partner expands due to the relaxation of their incentive compatibility (IC) constraints. The latter effect arises solely from the presence of two household members. The consideration of the optimal marginal tax rate for the under-provision of the household public good is amplified by a factor of $(1+E)$. In our framework, the marginal tax rate for the top earner is positive for Pigouvian consideration, which is different from the conventional result showing a zero marginal tax rate at the top.

In the case without tagging, the optimal marginal tax rates are characterized by the weighted average of the estimates in the $ABC + D(1+E)$ form for males and females who earn the same amount of income, regardless of whether they are part of a married couple. The weights are based on the distribution ratios of male and female spouses who earn the same amount of income. To the best of our knowledge, this study is the first to analyze optimal nonlinear income taxation that applies the same marginal tax rate to individuals earning the same amount of income, even when they belong to different households.

This study conducts a quantitative analysis using US wage data to further investigate the properties of optimal marginal tax rates. For comparison with the marginal tax rates for non-cooperative couples, we also provide quantitative marginal tax rates for cooperative couples providing the household public good at an efficient level. The comparison reveals that the marginal tax rates for non-cooperative couples are higher at all income levels compared with those for cooperative couples. The difference arises because the Pigouvian consideration imposed on the income taxes for non-cooperative couples raises the marginal tax rates. This result suggests that the optimal marginal tax rates provided in previous studies, which disregard non-cooperative behavior, may have been underestimated.

Moreover, we investigate how marginal tax rates should be differentiated between spouses when their housework productivities or wage rates differ. In the former case, the marginal tax rate for the spouse with higher housework productivity is higher than for the partner, incentivizing that spouse to contribute more toward housework, thereby enhancing the marital gains from the division of labor within the household. In the latter case, our quantitative analysis shows that a higher marginal tax rate is imposed on the spouse with the higher wage rate compared with the partner, to address equity considerations.

As an extensive and general case, this study considers the model in which both cooperative and non-cooperative couples exist. Furthermore, we introduce housework-specific commodities for household production and analyze the implementation of linear taxes and subsidies on these commodities. The subsidies for the consumption of housework-specific commodities are optimal under nonlinear income taxation.

The remainder of this paper is organized as follows. Section 2 describes the model. Section 3 analyzes the optimal taxation for non-cooperative couples, including a quantitative analysis. Section 4 explores the two extended models. Finally, Section 5 concludes the paper.

Related Literature

This study is related to the literature on couples' non-cooperative behavior. The traditional framework concerning the decision-making structure of households treats a household as a single decision-making agent, known as the "unitary" approach initiated by Samuelson (1956) and Becker (1965). However, the lack of empirical evidence for the unitary model leads to the development of the "collective" approach by Apps and Rees (1988) and Chiappori (1988, 1992), which incorporates bargaining power between spouses and assumes that households achieve the Pareto-efficient allocation. Both the unitary and collective approaches assume that intra-household behavior is efficient, but recent studies have increasingly employed a non-cooperative model to explain inefficient allocation, as in, for example, Konrad and Lommerud (1995), Cigno (2012), Cornes et al. (2012), and Gobbi (2018).[5] Empirical evidence supports the non-cooperative couple model (e.g., Del Boca and Flinn, 2012) and economic experiments (e.g., Ashraf, 2009; Cochard et al., 2016).

There is a growing body of literature analyzing optimal nonlinear income taxation for families and couples. Almost all the studies adopt family decision-making to achieve an efficient resource allocation, either a unitary or collective model. For example, Balestrino et al. (2002), Kleven et al. (2009), Cremer et al. (2012), Bastani et al. (2020), Ho and Pavoni (2020), Kurnaz (2021), and Golosov and Krasikov (2023) assume a unitary model; Schroyen (2003), Frankel (2014), Cremer et al. (2016), Gayle and Shephard (2019), Komura et al. (2021), Bierbrauer et al. (2023), and Alves et al. (2022) employ a collective model. Among these studies, Balestrino et al. (2002), Schroyen (2003), Kleven et al. (2009), Gayle and Shephard (2019), Ho and Pavoni (2020), and Kurnaz (2021) introduce household public goods to their models. However, their models do not result in under-provision of household public goods.

Meier and Rainer (2015) and Obara and Ogawa (2024) examine the optimal gender-based income tax structure in the models with under-provision of the household public goods due to non-cooperative behavior. However, they focus on linear income taxation to explore the implications of Ramsey taxation. Itaya et al. (2002) analyze private provision of public goods under optimal nonlinear income taxation. The crucial difference from our model is that they consider discrete types of agents and assume that the contribution to the public good is the provision of a portion of one's income, which is interpreted as donations or charity. Conversely, our model considers the Mirrleesian framework with continuous types of agents and assumes that the contribution is the provision of housework or childcare time, emphasizing the importance of the time provided by both spouses as demonstrated by Del Boca et al. (2014) and Lundborg et al. (2014). Due to these differences between the models, raising the income tax rate

[5]See related studies on non-cooperative models, including Lechene and Preston (2011), Doepke and Tertilt (2019), and Heath and Tan (2020).

increases the contribution in our model, whereas it decreases the contribution in Itaya et al. (2002).

2 Model

2.1 Preferences and Household Production

We introduce non-cooperative behavior of couples to the standard optimal income tax framework of Mirrlees (1971). A household comprises two spouses, denoted by a and b, where spouse a is of gender $\underline{A}$ and spouse b is of gender $\underline{B}$. We assume that the government's tax policy does not change the gender distribution of individuals. Each spouse non-cooperatively provides housework/childcare time for the household public good. To focus on non-cooperative behavior of the couples, this study assumes that each spouse also non-cooperatively determines their own consumption and labor supply. The utility function of each spouse takes the following form:

$$u_i(x_i, l_i, g_i) = v_i(x_i) + h_i\left(1 - l_i - g_i\right) + q, \quad i = a, b, \tag{1}$$

where x_i represents the consumption of a private good, l_i is a labor supply, g_i is the housework time, and q is the quantity of the household public good. The amounts of l_i and g_i can be interpreted as levels of effort. Hereafter, the prime "$\,'\,$" denotes the first-order derivative and the double prime "$\,''\,$" the second-order derivative. The sub-utility functions v_i and h_i satisfy $v_i' > 0 > v_i''$ and $h_i' > 0 > h_i''$.

To simplify the analysis, production of the household public good takes an additive separable form in terms of the contributions of the two spouses:

$$q = q_a(g_a) + q_b(g_b), \tag{2}$$

which satisfies $q_i' > 0 > q_i''$. The case can also be interpreted as the spouses providing different public goods, that is, separate spheres (Lundberg and Pollak, 1995).

Each spouse has an independent budget constraint and her/his income is spent on her/his own consumption, as in, for example, Konrad and Lommerud (1995), Meier and Rainer (2015), and Heath and Tan (2020).[6] This setting is supported by substantial evidence from studies such as Kenney (2006), Pahl (2008), and Lauer and Yodanis (2014). Each spouse's budget constraint is

$$x_i = y_i - T_i(y_i), \quad i = a, b, \tag{3}$$

where y_i is labor income and $T_i(y_i)$ is a tax function depending on spouse i's income. The income taxation is based on individual units, as discussed in the Introduction. Here, the government tags spouses a and b so as to impose different tax schedules for the genders. The case without tagging is analyzed in Section 3.3.2.

[6] Theoretical literature introducing the separate budget constraint also includes Lundberg and Pollak (1993), Anderberg (2007), Lechene and Preston (2011), and Doepke and Tertilt (2019).

The individual with ability w_i earns her/his income $y_i(= w_i l_i)$, with labor denoted by $y_i/w_i(= l_i)$. Considering this, (2), and (3), the utility function of spouse i can be rewritten as

$$u_i = v_i\left(y_i - T_i(y_i)\right) + h_i\left(1 - \frac{y_i}{w_i} - g_i\right) + q_i(g_i) + q_j(g_j), \quad i, j = a, b,\ i \neq j. \tag{4}$$

In our framework, as the government cannot directly observe housework time g_i, it indirectly manipulates unobservable housework time through the variation of observable income y_i. For indirect manipulation, we disaggregate the individual optimization process into two stages and express g_i as the function of y_i, following the process by Mirrlees (1976) and Jacobs and Boadway (2014).[7] First, the spouses decide non-cooperatively on their labor supply and private consumption. Next, they also non-cooperatively make decisions about their time devoted to the household public good.

2.2 Housework Time

Each spouse non-cooperatively decides her/his housework time, taking the partner's housework time as given. The first-order condition (FOC) for maximizing (4) with respect to g_i is

$$g_i : 0 = -h_i'\left(1 - \frac{y_i}{w_i} - g_i\right) + q_i'(g_i), \quad i = a, b. \tag{5}$$

As each spouse does not care about the effects of their own contribution to household production on their partner, the household public good is inefficiently under-provided. (5) leads to the following solution and properties:

$$g_i = g_i(w_i, y_i), \quad i = a, b, \tag{6}$$

which satisfies

$$\dot{g}_i\left(\equiv \frac{\partial g_i}{\partial w_i}\right) = \frac{y_i h_i''}{\left(h_i'' + q_i''\right) w_i^2} > 0, \quad i = a, b, \tag{7}$$

$$g_{iy_i}\left(\equiv \frac{\partial g_i}{\partial y_i}\right) = -\frac{h_i''}{\left(h_i'' + q_i''\right) w_i} < 0, \quad i = a, b. \tag{8}$$

The contribution of spouse i is independent of that of the partner because of the additive separable form of the production of the household public good. (7) shows that individuals with higher wage rates spend more time on housework as the nature of the utility function balances the consumption for private and public goods. From (8), we confirm the substitutability of labor supply and housework time that is essential for this study's analysis.

[7] Stiglitz (1982), Findeisen and Sachs (2017), and Obara (2019) also treat an unobservable variable for the government as the function of observable variables.

2.3 Labor Supply

Labor income y_i (and therefore labor supply $l_i(= y_i/w_i)$) is determined by considering (6). Substituting (6) into (4), maximizing the resulting utility function with respect to y_i, and utilizing (5), we obtain the following FOC:

$$y_i : 0 = v_i' \left(y_i - T_i(y_i)\right)\left(1 - T_i'(y_i)\right) - h_i' \left(1 - \frac{y_i}{w_i} - g_i(w_i, y_i)\right) \frac{1}{w_i}, \quad i = a, b, \tag{9}$$

where $T_i'(y_i) \equiv dT_i/dy_i$. (9) implies that

$$y_i = y_i(w_i), \quad i = a, b. \tag{10}$$

The spouses' income is generally presumed to depend on each other with the private provision of household public goods. However, each spouse's income, shown in (10), depends solely on her/his own wage. This is because of the additive separable form of home production and the independent budget constraints of the spouses. This simplification facilitates the analysis and allows us to address the under-provision of the household public good in an optimal taxation framework.

From (6) and (10),

$$g_i = g_i(w_i, y_i(w_i)), \quad i = a, b. \tag{11}$$

The government knows this functional form and indirectly manipulates g_i through assignment of y_i in scheduling the optimal nonlinear income taxes.

2.4 Assortative Mating

Abilities w_a and w_b correspond with the wages of spouses a and b, respectively, given that aggregate production is linear in labor, and are distributed according to the cumulative distribution function $\widetilde{\Pi}(w_a, w_b)$ for $(w_a, w_b) \in W_a \times W_b = [\underline{w}_a, \overline{w}_a] \times [\underline{w}_b, \overline{w}_b]$, where $0 \leq \underline{w}_i < \overline{w}_i < \infty$ for $i = a, b$. This implies the multidimensionality that arises from dealing with couples.

To avoid multidimensionality, we assume the assortative mating of couples that allows us to reduce the household's multidimensional type (w_a, w_b) into unidimensionality, as in Cremer et al., (2016). This assumption aligns with recent trends in assortative mating, which have led to growing inequality among households (Carbone and Cahn, 2014; Eika et al., 2019).

The assumption of assortative mating allows us to express the ability of a spouse as a power function of the ability of the other spouse. In this study, without loss of generality, we consider the power function of w_b depending on w_a, such that $w_b = \alpha(w_a) \geq 0$ for $w_a \in W_a$, which is assumed to be continuously differentiable and strictly increasing, i.e., $\alpha'(w_a) > 0$ for $w_a \in W_a$. Let us denote w_a as w, so that $w_b = \alpha(w)$. For the brevity of the expressions of the function and use of notations, we express w_a and w_b as the function of w as follows:

$$w_a = w_a(w) \equiv w, \quad w_b = w_b(w) \equiv \alpha(w), \quad \text{for } w \in W_a. \tag{12}$$

Let us define $w_i' \equiv \partial w_i/\partial w$ and note that $w_a' = 1$ and $w_b' = \alpha'$. Thus, abilities w_a and w_b are distributed according to the function $\Pi(w) \equiv \widetilde{\Pi}(w, \alpha(w))$. The probability density function, $\pi(w)\,(= d\Pi/dw)$, is assumed to be continuously differentiable and positive for $w \in W_a$. The density of w_a must be equal to that of w_b from the assortative mating of couples. (12) includes the perfect assortative mating that is given by

$$w_a(w) = w_b(w) = w, \quad \text{for } w \in W_a. \tag{13}$$

This case is treated in the quantitative analysis in Section 3.3.

Allowing for (12), the variables of the spouses can be expressed as the function of w, representing the indicator of the w-th household. Thus, (10) and (11) can be expressed, respectively, by

$$y_i = y_i(w), \quad i = a, b, \tag{14}$$

$$g_i = g_i(w, y_i(w)), \quad i = a, b. \tag{15}$$

Note that $y_i(w)$ and $g_i(w, y_i(w))$ includes the information of $w_i(w)$ of spouse i. Those reduced forms expressed as a function of w considerably contribute to the brevity of expressions of formulae, conditions, and definitions presented in subsequent analyses.

2.5 Government

Allowing for (12), (14), and (15), the individual utility function is given by[8]

$$\begin{aligned} u_i(w) = v_i(y_i(w) - T(y_i(w))) + h_i\left(1 - \frac{y_i(w)}{w_i(w)} - g_i(w, y_i(w))\right) \\ + q_i(g_i(w, y_i(w))) + q_j(g_j(w, y_j(w))), \quad i, j = a, b,\ i \neq j. \end{aligned} \tag{16}$$

Welfare criterion for the couples is defined by $\Phi(u_a, u_b)$ with $\Phi_{u_i}(\equiv \partial\Phi/\partial u_i) > 0$ and $\Phi_{u_i u_i}(\equiv \partial\Phi_{u_i}/\partial u_i) \leq 0$, which implies a non-negative aversion to inequality and Φ is independent of w. Then, social welfare is defined as the sums of all couples, that is,

$$\int_{\underline{w}}^{\overline{w}} \Phi(u_a(w), u_b(w))\pi(w)dw. \tag{17}$$

The government chooses the consumption-utility bundle intended for each household $\{y_a(w),\ y_b(w),\ u_a(w),\ u_b(w),\ w \in W_a\}$, or equivalently the tax schedule $T_i(\cdot)$, to maximize the social welfare subject to two types of constraints. The first is the government budget constraint, given by

$$\int_{\underline{w}}^{\overline{w}} (T_a(y_a(w)) + T_b(y_b(w)))\pi(w)dw \geq R, \tag{18}$$

where R is an exogenous revenue requirement. This constraint must be binding at the optimum, as utility increases with consumption.

[8]Individual preference in our model ensures that the strict-single crossing (Spence-Mirrlees) condition holds even in the presence of the household public good.

The second is the set of incentive compatibility constraints (IC constraints), which require that type-w agents choose the consumption-income bundle intended for them.[9] Under the assumption of assortative mating, the government knows that spouse a with m-th highest wage in gender $\underline{A}$ is paired with spouse b with m-th highest wage in gender $\underline{B}$. It also knows w_a and w_b for any m. We assume that individuals cannot mimic gender. What the government does not know is which couple has the m-th highest wage. Using (5) and (9), we derive the IC constraint for each spouse from (16) as follows:

$$\dot{u}_i(w) = h_i' \left(1 - \frac{y_i(w)}{w_i(w)} - g_i(w, y_i(w))\right) \frac{y_i(w) w_i'(w)}{(w_i(w))^2} + q_j'(g_j(w, y_j(w)))\dot{g}_j(w, y_j(w)) w_j'(w), \quad i, j = a, b,\ i \neq j, \tag{19}$$

where $\dot{u}_i$ denotes the derivatives with respect to the wage rate of spouse i ($\dot{u}_i \equiv du_i/dw$). The derivation of (19) is provided in Appendix A. The first term corresponds to the standard IC constraint form from previous studies. The second term is the new one and reflects the effects of the contribution of spouse j, who is the partner of spouse i, to the household public good. As spouse j does not consider the effects of her/his contribution on spouse i and spouse i takes spouse j's contribution as given, the second term remains in the IC constraint of spouse i. The second term is positive, as $q_j' > 0$ and $\dot{g}_j > 0$ from (7). Therefore, $\dot{u}_i > 0$ holds. This shows that the higher the value of the second term, the less likely an individual is to mimic, which relaxes the restrictions on income taxes.

3 Optimal Taxation

The government must implement the desired allocations for each individual by using nonlinear labor income taxes to maximize the social welfare function (17) subject to the budget constraint (18) and the IC constraints (19). Before presenting the optimal income tax rate, we define the following elasticities (see Appendix B):

$$\varepsilon_{lT'}^{ci} \equiv \frac{1 - T_i'}{l_i} \left.\frac{\partial l_i}{\partial(1 - T_i')}\right|_{u_i = \text{constant}} = -\frac{h_i'}{\left[(1 - T_i')^2 w_i^2 v_i'' + (1 + w_i g_{iy_i}) h_i''\right] l_i} > 0, \quad i = a, b, \tag{20}$$

$$\varepsilon_{lT'}^{ui} \equiv \frac{1 - T_i'}{l_i} \frac{\partial l_i}{\partial(1 - T_i')} = -\frac{h_i' + (1 - T_i')^2 w_i^2 l_i v_i''}{\left[(1 - T_i')^2 w_i^2 v_i'' + (1 + w_i g_{iy_i}) h_i''\right] l_i}, \quad i = a, b, \tag{21}$$

$$\varepsilon_{gT'}^{ci} \equiv -\frac{1 - T_i'}{g_i} \left.\frac{\partial g_i}{\partial(1 - T_i')}\right|_{u_i = \text{constant}} = \frac{w_i g_{iy_i} h_i'}{\left[(1 - T_i')^2 w_i^2 v_i'' + (1 + w_i g_{iy_i}) h_i''\right] g_i} > 0, \quad i = a, b, \tag{22}$$

[9] We assume that $w \longmapsto y_i(w)$ is continuous on $[\underline{w}, \overline{w}]$ and differentiable everywhere, except for an infinite number of ability levels, and that $w \longmapsto u_i(w)$ is differentiable. Hence, $w \longmapsto x_i(w)$ and $w \longmapsto g_i(w)$ are also continuous everywhere and differentiable almost everywhere.

$$\varepsilon^i_{q'y} \equiv \frac{y_i}{q'_i}\frac{\partial q'_i}{\partial y_i} = \frac{y_i q''_i g_{iy_i}}{q'_i} > 0, \quad i = a, b, \tag{23}$$

$$\varepsilon^i_{\dot{g}y} \equiv \frac{y_i}{\dot{g}_i}\frac{\partial \dot{g}_i}{\partial y_i} = 1 + \frac{(q''_i)^2 y_i g_{iy_i}}{h''_i + q''_i}\left[\frac{h'''_i}{(h''_i)^2} - \frac{q'''_i}{(q''_i)^2}\right], \quad i = a, b, \tag{24}$$

where "$'''$" in the superscript denotes the third-order derivatives. The elasticities $\varepsilon^{ci}_{lT'}$ and $\varepsilon^{ui}_{lT'}$ represent the compensated and uncompensated elasticities of labor supply with respect to the tax rate. Unlike standard elasticities, these elasticities also account for the effects on housework time, g_{iy_i}. $\varepsilon^{ci}_{gT'}$ is the compensated elasticity of housework time with respect to the tax rate, reflecting the Pigouvian consideration in the optimal marginal tax rate formula. $\varepsilon^i_{q'y}$ is the income elasticity of the marginal productivity of housework time, while $\varepsilon^i_{\dot{g}y}$ is the income elasticity of the marginal housework time, both of which appear in the term related to the relaxation of the partner's IC constraints in the formula for the optimal marginal tax rate. From (14) and (15), we can express the elasticities as functions of w: $\varepsilon^{ci}_{lT'}(w)$, $\varepsilon^{ui}_{lT'}(w)$, $\varepsilon^{ci}_{gT'}(w)$, $\varepsilon^i_{q'y}(w)$, and $\varepsilon^i_{\dot{g}y}(w)$. Let us make the following definition: $\beta_i \equiv g_i/l_i = w_i g_i/y_i$, which appears in the optimal marginal tax formula. From (10)–(12), we have $\beta_i(w) = w_i(w)g_i(w, y_i(w))/y_i(w)$.

3.1 Cooperative Couples: Benchmark

This section presents the formula for the optimal marginal tax rates in the cooperative case, where no under-provision of the household public good occurs, to compare with the non-cooperative case. In contrast to the non-cooperative case, in the cooperative setting, couples jointly decide on labor supply, housework time for household production, and private consumption to achieve efficient allocations within the household. Each household maximizes the sum of the spouses' utilities with equal weight:

$$\begin{aligned} u_a + u_b(\equiv u) = {} & v_a(x_a) + v_b(x_b) + h_a\left(1 - \frac{y_a}{w_a} - g_a\right) \\ & + h_b\left(1 - \frac{y_b}{w_b} - g_b\right) + 2q_a(g_a) + 2q_b(g_b). \end{aligned} \tag{25}$$

The last two terms, which capture the benefit of the household public good, are twice as large as the corresponding terms in the individual utility functions of non-cooperative couples. This is because cooperative couples consider the effects of the household public good on both spouses. The household's budget constraint is given by

$$x_a + x_b = y_a + y_b - T_a(y_a) - T_b(y_b).$$

The decision-making mechanism of the household assumes collective decision-making with equal bargaining power for the spouses, resulting in efficient allocations and no under-provision of the household public good. The social welfare function is defined by $\Phi(u)$, which implies that the government places equal weight on each spouse within the household. As both the household's and the government's weights are equal across the spouses, the optimal income taxation does not need to account for dissonance within households, unlike the models of Apps and Rees (1988) and Alves et al. (2024). Let us denote $\Phi' \equiv d\Phi/du$. The marginal tax rates are given by the following lemma (Appendix C).

Lemma 1. *In the cooperative case, the optimal marginal tax rate is given by*

$$\frac{T_i'(w)}{1-T_i'(w)} = A_i^c(w)B_i^c(w)C_i^c(w), \tag{26}$$

where

$$A_i^c(w) \equiv \frac{1+\varepsilon_{lT'}^{ui}(w)}{\varepsilon_{lT'}^{ci}(w)}, \tag{27}$$

$$B_i^c(w) \equiv \int_w^{\overline{w}} \left(\frac{1}{v_i'(x_i(s))} - \frac{\Phi'(u_a(s)+u_b(s))}{\lambda}\right)\pi(s)ds \left(\frac{w_i'(w)}{1-\Pi(w)}\right), \tag{28}$$

$$C_i^c(w) \equiv \frac{(1-\Pi(w))\, v_i'(x_i(w))}{\pi(w)w_i(w)}, \tag{29}$$

for $i = a, b$.

The optimal marginal tax rate follows the standard Mirrlees–Diamond ABC formula derived for a single household case, incorporating the effects of taxation on tax-induced deadweight loss and and revenue effects (mechanical effects). However, it is important to note that the elasticities in the formula provided in Lemma 1 also include the effects on housework time, g_{iy_i}. The ABC form appears as part of the optimal marginal tax formula for non-cooperative couples, as shown later.

3.2 Non-Cooperative Couples

3.2.1 Tagging

This section provides the optimal marginal tax rate expression for the non-cooperative couples in the case with tagging genders; that is, so-called gender-based taxation is applied. We will examine the case without tagging in the next section. The optimal marginal tax rates are presented in the following proposition (Appendix D).

Proposition 1. *(With Tagging) The optimal marginal tax rates on the non-cooperative couples in the case with tagging satisfies*

$$\frac{T_i'(w)}{1-T_i'(w)} = A_i^n(w)B_i^n(w)C_i^n(w) + D_i^n(w)(1+E_i^n(w)), \tag{30}$$

where

$$A_i^n(w) \equiv \frac{1+\varepsilon_{lT'}^{ui}(w)}{\varepsilon_{lT'}^{ci}(w)}, \tag{31}$$

$$B_i^n(w) \equiv \int_w^{\overline{w}} \left(\frac{1}{v_i'(x_i(s))} - \frac{\Phi_{u_i}(u_a(s), u_b(s))}{\lambda} \right) \pi(s) ds \left(\frac{w_i'(w)}{1 - \Pi(w)} \right), \tag{32}$$

$$C_i^n(w) \equiv \frac{(1 - \Pi(w))\, v_i'(x_i(w))}{w_i(w)\pi(w)}, \tag{33}$$

$$D_i^n(w) \equiv \beta_i(w) \frac{\varepsilon_{gT'}^{ci}(w)}{\varepsilon_{lT'}^{ci}(w)} \frac{v_i'(x_i(w))}{v_j'(x_j(w))}, \tag{34}$$

$$E_i^n(w) \equiv B_j^n(w) C_j^n(w) \frac{w_j(w) w_i'(w)}{w_i(w) w_j'(w)} \left(\varepsilon_{q'y}^i(w) + \varepsilon_{\dot{g}y}^i(w) \right), \tag{35}$$

for $i, j = a, b$ *and* $i \neq j$.

The optimal marginal tax rate formula comprises two terms. The first term is the well-known Mirrlees–Diamond ABC form, as presented in previous studies. The second term is the Pigouvian $D(1+E)$ form, a new term that arises due to the under-provision of the household public good resulting from non-cooperative behavior. This is confirmed by Lemma 1, which demonstrates that when the household public good is efficiently provided, only the ABC form characterizes the optimal marginal tax rate. It should be noted that the optimal tax rate for a spouse also depends on the behavior of her/his partner: $v_j'(x_j)$ appears in the term D_i^n, while B_j^n and C_j^n are in E_i^n.

The Pigouvian $D_i^n(1 + E_i^n)$ form can be further decomposed into the terms D_i^n and $D_i^n E_i^n$. The term D_i^n captures the effects of improving the under-provision of the household public good from an efficiency perspective. As $\varepsilon_{gT'}^{ci}$ increases, income taxation becomes more effective at inducing additional housework to address this under-provision, thereby raising the marginal tax rate. The term $D_i^n E_i^n$ reflects the impact of enlarging the income tax flexibility through the change in the household public good. The term E_i^n indicates the extent to which the flexibility of the income taxes on the partner expands due to the relaxation of the partner's IC constraints. As noted in (19), an increase in the spouse's contribution weakens the incentive for their partner to mimic. Relaxing the restrictions on income taxes enables the government to increase their adjustability. The consideration of the optimal marginal tax rate for the under-provision of the household public good is amplified by a factor of $(1 + E)$ if $E > 0$. Our quantitative analysis shows that $E > 0$. The term E_i^n includes the mechanical effect of the partner, expressed as $\int_w^{\overline{w}} \left(\frac{1}{v_j'} - \frac{\Phi_{u_j}}{\lambda} \right) \pi ds$, which reflects the extent to which the flexibility of the income taxes on the partner increases.

To further deepen the interpretation of the optimal income tax within our framework, we consider two cases. First, we take the case in which the optimal tax rate equals the term D_i^n. Consider the top earners. It is clear that the mechanical effects become zero, that is, $\int_w^{\overline{w}} \left(\frac{1}{v_i'} - \frac{\Phi_{u_i}}{\lambda} \right) \pi ds = 0$ at $w = \overline{w}$ for $i = a, b$. Thus, we immediately see that $A_i^n(\overline{w}) B_i^n(\overline{w}) C_i^n(\overline{w}) = 0$. As $B_j^n(\overline{w}) C_j^n(\overline{w}) = 0$ as well, $E_i^n(\overline{w}) = 0$ from (35). Consequently, (30) leads to

$$\frac{T_i'(\overline{w})}{1 - T_i'(\overline{w})} = D_i^n(\overline{w}) = \beta_i(\overline{w}) \frac{\varepsilon_{gT'}^{ci}(\overline{w})}{\varepsilon_{lT'}^{ci}(\overline{w})} \frac{v_i'(x_i(\overline{w}))}{v_j'(x_j(\overline{w}))} > 0, \quad i, j = a, b, \ i \neq j. \tag{36}$$

In contrast to conventional results, the marginal tax rate is not zero but positive for the top earner.

We now provide a somewhat special case, which is suggestive and therefore worth examining in more detail. Assume that the utility is quasi-linear, that is, $v_i' = 1$, and the Bentham criterion, $\Phi_{u_i} = 1$. Under these conditions, $\lambda = 1$ holds.[10] Therefore, we see that the mechanical effects in B_j^n become zero. Following the above process, we immediately see that $A_i^n(w)B_i^n(w)C_i^n(w) = 0$ and $E_i^n(w) = 0$. Therefore, we obtain

$$\frac{T_i'(w)}{1 - T_i'(w)} = D_i^n(w)\big|_{v_a'=v_b'=1} = \beta_i(w)\frac{\varepsilon_{gT'}^{ci}(w)}{\varepsilon_{lT'}^{ci}(w)} > 0, \quad i = a, b. \tag{37}$$

This equation implies that optimal income taxes are designed solely to correct the under-provision of the household public good for efficiency, as considering equity and the IC constraints is not necessary.

Second, we consider the case in which one spouse does not contribute to the household public good at all. In this case, their marginal tax rate formula becomes the standard Mirrlees formula. Without loss of generality, we assume that spouse a does not contribute to the household public good.

Corollary *If spouse a does not contribute to household production at all, the optimal marginal tax rates satisfy*[11]

$$\frac{T_a'(w)}{1 - T_a'(w)} = A_a^n(w)B_a^n(w)C_a^n(w), \tag{38}$$

$$\frac{T_b'(w)}{1 - T_b'(w)} = A_b^n(w)B_b^n(w)C_b^n(w) + D_b^n(w)(1 + E_b^n(w)). \tag{39}$$

See Appendix E for the proof of Corollary 1. As the optimal income tax structure does not need to account for the effect of spouse a's contribution on the IC constraint of spouse b, the Pigouvian $D(1 + E)$ form disappears in the optimal marginal tax formula for spouse a. If $D_b^n(1 + E_b^n) > 0$, the tax rate for spouse b increases by an additional amount $D_b^n(1 + E_b^n)$ on top of the values from the term $A_b^n B_b^n C_b^n$. In other words, a higher tax rate may be imposed on the contributor of the household public good.

3.2.2 No Tagging

The case without tagging is treated in this section, where the same marginal tax rate is applied to individuals with the same amount of income, regardless of gender and

[10] This fact is confirmed from (109)–(112) in Appendix D.

[11] As spouse a does not provide the household public goods, her/his elasticities are

$$\varepsilon_{lT'}^{ca} = -\frac{h_a'}{\left[(1 - T_a')^2 w_a^2 v_a'' + h_a''\right] l_a}, \quad \text{and} \quad \varepsilon_{lT'}^{ua} = -\frac{h_a' + (1 - T_a')^2 w_a^2 v_a'' l_a}{\left[(1 - T_a')^2 w_a^2 v_a'' + h_a''\right] l_a}.$$

if they belong to different couples. This is a realistic and commonly employed tax system used in many countries. This concept of applying the same marginal tax rate to individuals in different couples is novel, even when household public goods are not under-provided.

As both utility and household production functions are additively separable, individuals a and b with the same abilities always receive the same income, regardless of whether they belong to the same couple. In other words, $y_a(w_a(w)) = y_b(w_b(\widetilde{w}))$ if and only if $w_a(w) = w_b(\widetilde{w})$. Noting that $\iota \equiv w_a(\iota)$, let us define the function of the wage rate of the partner of individual b that equals to the wage rate ι of individual a as $\widetilde{w}(\iota)$. Then $\iota \equiv w_a(\iota) = w_b(\widetilde{w}(\iota))$. This enables us to make the analysis in the one-dimensional parameter ι. Let us define the ratio of distributions for individual a with ability $w_a(\iota)$ and for individual b with ability $w_b(\widetilde{w}(\iota))$, respectively, as

$$\tau(\iota) \equiv \frac{\pi(\iota)}{\pi(\iota) + \pi(\widetilde{w}(\iota))}, \quad 1 - \tau(\iota) \equiv \frac{\pi(\widetilde{w}(\iota))}{\pi(\iota) + \pi(\widetilde{w}(\iota))}.$$

In this section, we omit the index i in the income tax function and represent it simply as $T(y_i)$. The optimal marginal income tax rate in the case without tagging is given in the following proposition (Appendix F).

Proposition 2. *(Without Tagging) The optimal marginal tax rate on the non-cooperative couples in the case without tagging satisfies that*

$$\begin{aligned} \frac{T'(\iota)}{1 - T'(\iota)} &= \tau(\iota)[A_a^n(\iota)B_a^n(\iota)C_a^n(\iota) + D_a^n(\iota)(1 + E_a^n(\iota))] \\ &\quad + (1 - \tau(\iota))\,[A_b^n(\widetilde{w}(\iota))B_b^n(\widetilde{w}(\iota))C_b^n(\widetilde{w}(\iota)) + D_b^n(\widetilde{w}(\iota))(1 + E_b^n(\widetilde{w}(\iota)))]. \end{aligned} \tag{40}$$

The optimal marginal tax rate is expressed by the weighted average of the $ABC + D(1 + E)$ formula for individual a with ability $w_a(\iota)$ and that for individual b with ability $w_b(\widetilde{w}(\iota))(= \iota)$, with the weight being the ratio of distributions for individual a with ability $w_a(\iota)$ and for individual b with ability $w_b(\widetilde{w}(\iota))$. The marginal tax rate without tagging always lies between the estimates of $A_i^n B_i^n C_i^n + D_i^n(1 + E_i^n)$ for individuals a and b. Therefore, these rates must be adjusted based on the gender with the larger distribution.

From the definitions of A_i^n, B_i^n, C_i^n, D_i^n, and E_i^n, we see that the marginal tax rates for the individuals earning a certain income level depend on the properties of four types of taxpayers: individual a with ability $w_a(\iota)$, individual b with ability $w_b(\widetilde{w}(\iota))$ (which equals $w_a(\iota)$), and their respective partners, as explained below Proposition 1.

3.3 Quantitative Analysis

In this section, we compute the optimal marginal tax rates derived in Proposition 1, using US wage data. To examine the properties of the marginal tax rates for non-cooperative couples, we compare them with the marginal tax rates for cooperative couples derived in Lemma 1. The quantitative analysis of optimal nonlinear income taxation essentially relies on a discrete model due to dataset limitations. Thus, an equal number of spouses a and b are assumed to exist to ensure all individuals are in married couples.

3.3.1 Numerical Specification

The social welfare function is specified by CRRA under the sum of each spouse's utility, as follows:

$$\Phi(u_a, u_b) = \log(u), \quad \text{where } u = u_a + u_b.$$

The government assigns equal weights to the spouses in each household, which is consistent with the household utility for cooperating couples presented in Section 3.2. In this case, $\Phi_{u_a} = \Phi_{u_b} = 1/u$. The logarithmic utility function is based on estimates of the curvature of utility functions, consistent with labor supply responses (Chetty, 2006).

The household production technology is specified as

$$q_i(g_i) = f_i \log(g_i), \quad i = a, b, \tag{41}$$

where f_i denotes the gender-specific productivity and is assumed to be 2.5. Under this equation, the elasticity of housework time is one, that is, $\varepsilon^i_{gT'} = 1$ for $i = a, b$.

The sub-utility function with respect to $l_i + g_i$ is given by

$$h(1 - l_i - g_i) = -\frac{\varkappa}{1+\zeta_i}(l_i + g_i)^{1+\zeta_i}, \quad i = a, b,$$

with $\varkappa = 2.55$. Based on the estimates from Chetty et al. (2011), we set $\zeta_i = 2$, implying that the elasticity of labor supply is 0.5. In this case, the elasticity of labor supply $\varepsilon^i_{lT'}$ can be rewritten as

$$\varepsilon^i_{lT'} = \frac{1}{\zeta_i}\frac{l_i + g_i}{l_i} + \frac{g_i}{l_i} = \frac{1}{\zeta_i}(1 + \beta_i) + \beta_i, \quad i = a, b. \tag{42}$$

The sub-utility function with respect to x_i is specified by

$$v_i(x_i) = \frac{x_i^{1-\xi} - 1}{1 - \xi}, \quad i = a, b,$$

with $\xi = 0.4$.

To perform a quantitative analysis, the ability distribution needs to be specified. In this study, we use the same dataset used in Mankiw et al. (2009) and follow their calibration procedure closely. They use the March wave of the Current Population Survey (CPS) to parameterize the US wage distribution. We employ wage data for 2007 using a lognormal parameterization with parameters $(\mu, \sigma) = (2.757,\ 0.5611)$ up to a wage level of \$42.50. For the lognormal-Pareto distribution, we append a Pareto tail with a parameter setting of 2, adapted from Saez (2001), above the wage of \$42.50. The Pareto tail is scaled so that the resulting lognormal-Pareto distribution is continuous and integrates to one. In accordance with most earlier literature, we assume that a mass of disabled workers are earning a wage of \$0.01, with the fraction of disabled agents in the population assumed to be 5%. The wages should be interpreted as hourly wages.

3.3.2 Main Results

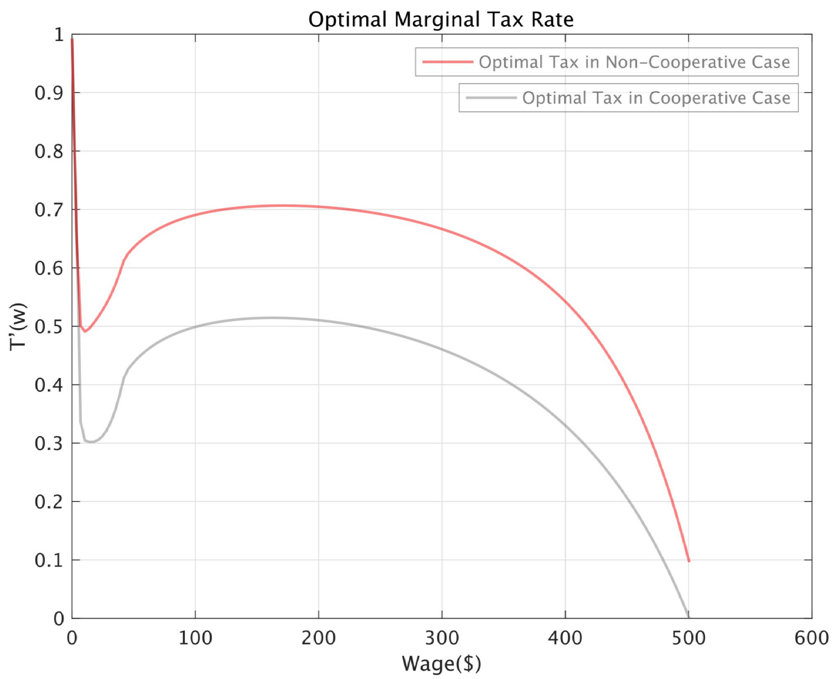

Figure 1: The red line depicts the values for $T'/(1-T')$ that satisfying Eq. (30) in Proposition 1 considering non-cooperative behavior of the couple. The gray line depicts the values for $T'/(1-T')$ that satisfy Eq. (26) in Lemma 1 considering cooperative behavior of the couple. As spouses a and b in each household are identical, the same marginal tax rates are applied to the spouses in each household.

Now, we numerically compute the marginal income tax rates based on the functional specifications and US wage data described above. First, we consider identical spouses for each household, so that the same marginal tax rate is applied to both spouses. Figure 1 depicts the optimal marginal tax rates for non-cooperative and cooperative cases: the value for the marginal tax rate in the non-cooperative case, which satisfies the formula in Proposition 1, is depicted by the red line, and the value for the cooperative case, which satisfies the formula in Lemma 1, is shown by the gray line. The marginal tax rates decrease at low-income levels (i.e., the high tax rates at the bottom correspond to the phasing-out of the guaranteed income level), and then increase until the middle-income level. This shape is consistent with the findings of Diamond (1998) and Saez (2001). Beyond this point, the optimal tax rates decrease again at higher income levels to incentivize high-income earners to work more. We

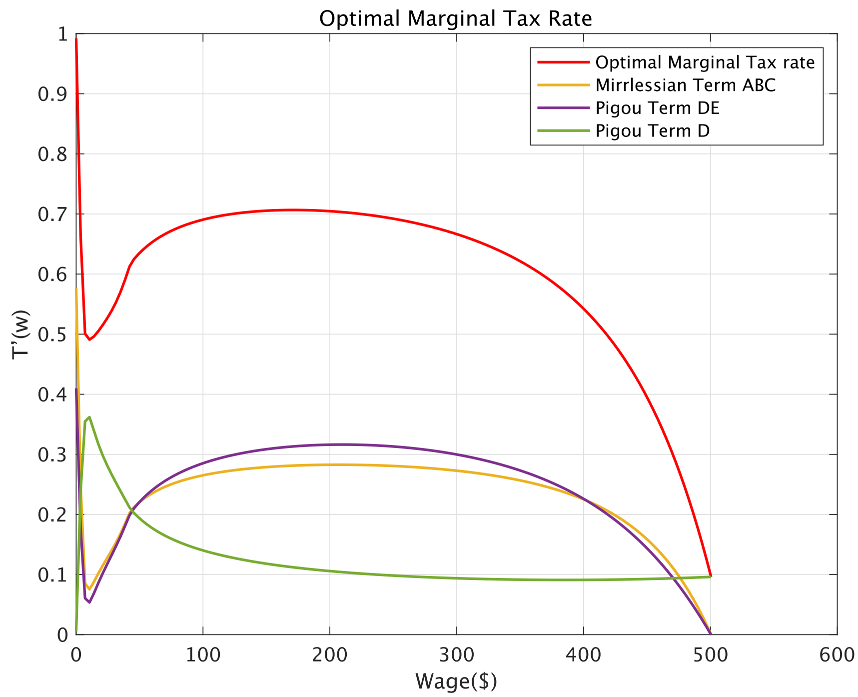

Figure 2: The red line depicts the values for $T'/(1-T')$ that satisfying Eq. (30) in Proposition 1 considering non-cooperative behavior of the couple. The other three lines decompose $T'/(1-T')$ into three terms. The Mirrlessian term ABC satisfying (31) multiplied by (32) and (33) is depicted by the yellow line, the Pigouvian term D satisfying (34) by the green line, and the Pigouvian term DE satisfying (34) multiplied by (35) by the purple line. As spouses a and b in each household are identical, the same marginal tax rates are applied to the spouses in each household.

observe that the marginal tax rates for non-cooperative couples are higher at all income levels compared with those for cooperative couples. This difference in tax rates is due to Pigouvian considerations, which drive up the marginal tax rates for non-cooperative couples. The optimal tax rate remains positive at the top, consistent with our theoretical result. As previous studies analyze the optimal income tax problem without accounting for noncooperative behavior, they are considered to underestimate the optimal marginal tax rates.

To provide a more detailed explanation of the marginal tax rates for non-cooperative couples, we decompose them into three terms. In Figure 2, the value of Mirrlees's ABC form is shown with the yellow line, the term D with the green line, the term DE with the purple line, and the optimal marginal tax rate with the red line. The decomposition results in several findings. First, because the Pigouvian terms D and DE are both positive, the curve for the optimal marginal tax rate lies above that of Mirrlees's ABC form by the sum of the two terms. Second, the marginal tax rate equals the value of the Pigouvian term D for top earners, which is consistent with equation (36). Third, the value of term D is relatively low for the spouse with higher ability. This is because, for individuals with higher wage rates, the increase in income from more labor supply is so large that the welfare improvement from the income increase outweighs

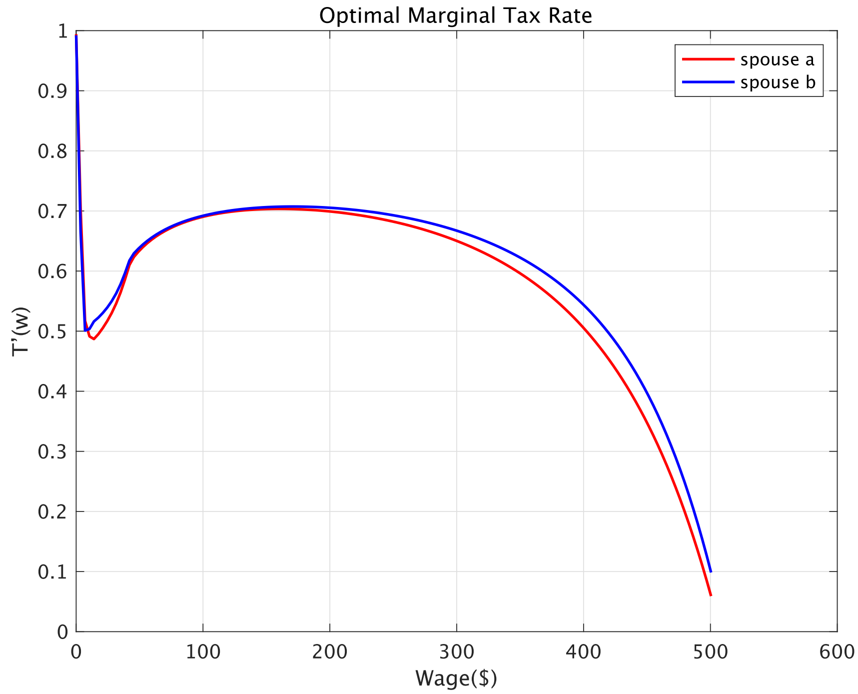

Figure 3: The red line depicts the values for $T'_a/(1 - T'_a)$ on spouse a that satisfying Eq. (30) in Proposition 1, while the blue line depicts the values for $T'_b/(1 - T'_b)$ on spouse b. This is the case where spouse b's productivity of household production is higher than spouse as productivity.

the welfare improvement from the increase in household public goods due to more housework time. Fourth, in Figure 3, the term E is positive, as both terms D and DE are positive. This shows that the welfare effects of improving the under-provision of household public goods are strengthened by the magnitude of $(1 + E)$, which is due to the expansion effects of tax flexibility.

Next, we consider asymmetric non-cooperative spouses with respect to the gender-specific productivities of housework or the abilities in the external labor market. Figure 3 illustrates the case where spouse b's productivity in household production is higher than that of spouse a, specifically with $f_a = 1.5$ and $f_b = 2.5$. Spouse b's marginal tax rate is higher than that of spouse a. The optimal income tax system provides spouse b with a greater incentive to engage in more housework and spouse a with a larger incentive to increase their labor supply in the external market, thereby enhancing efficiency in each household. The tax rate difference is scheduled to enhance the gain from the marital division of labor.

Figure 4 considers the case where spouse b's ability is half that of spouse a at lower income levels, and the wage differential decreases as productivity increases, approaching zero at higher income levels. The finding that a higher marginal tax rate should be imposed on the spouse with the higher wage rate contrasts the result of Meier and Rainer (2015), who show that, in the linear income tax case with a representative spouse, a higher tax rate is imposed on the spouse with the lower wage rate. The driv-

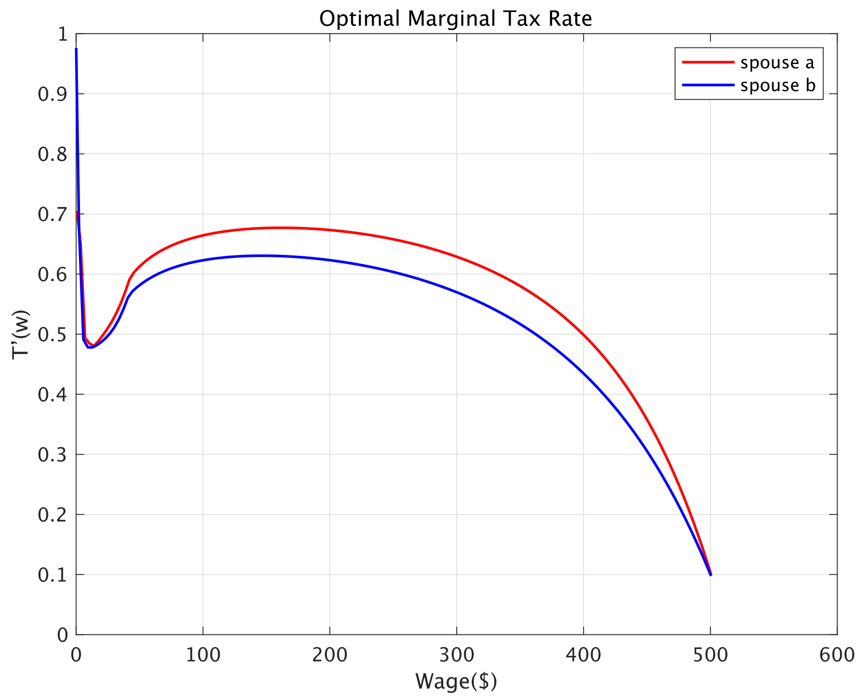

Figure 4: The red line depicts the values for $T_a'/(1-T_a')$ on spouse a that satisfy Eq. (30) in Proposition 1, while the blue line depicts the values for $T_b'/(1-T_b')$ on spouse b. This is the case where spouse b's ability is twice less than spouse a's ability at the bottom and then the wage differentials decrease with productivity and that gets close to zero at the top.

ing force behind our result is a strong equity consideration in the case of heterogeneous individuals.

4 Extensions

4.1 Both Cooperative and Non-cooperative Couples Exist

This section considers the model where both cooperative and non-cooperative couples exist. The government knows the ratio of the cooperative couples compared to the non-cooperative couples for each ability, but do not know which couple is cooperative or non-cooperative. We assume that taxation does not change the behavioral pattern of a couple in the sense that the couple becomes cooperative or non-cooperative.

Let denote the ratio of the non-cooperative couples with abilities $(w_a(w), w_b(w))$ as $\gamma(w)$, that is, that of the cooperative couples as $1-\gamma(w)$. To avoid confusion, in this section, we include symbol c in the superscript of all variables of the cooperative couples, and n in that of the non-cooperative couples. The government's objective function is assumed to simply be utilitarian:

$$\int_{\underline{w}}^{\overline{w}} [\gamma(w)(u_a^n(w)+u_b^n(w))+(1-\gamma(w))(u_a^c(w)+u_b^c(w))]\pi(w)dw. \tag{43}$$

The government budget constraint is given by

$$\int_{\underline{w}}^{\overline{w}} \left[\gamma(w)(T_a(y_a^n(w)) + T_b(y_b^n(w))) + (1-\gamma(w))(T_a(y_a^c(w)) + T_b(y_b^c(w)))\right] \pi(w)dw \geq R. \tag{44}$$

Let $\widehat{w}$ be the ability of spouse i in the cooperative couple that has the same income as that in the non-cooperative couple with w. Then, $y_i^n(w) = y_i^c(\widehat{w})$ for $i = a, b$. Here, $w \neq \widehat{w}$ in general, as the level of household public good provision differs between cooperative and noncooperative couples. Based on the equation $y_i^n(w) = y_i^c(\widehat{w})$, we introduce a function $\widehat{w}(w)$ that allows us to reduce the analysis to the one-dimensional parameter w. Then, $y_i^n(w) = y_i^c(\widehat{w}(w))$. As the government cannot observe which couples are cooperative or uncooperative, it imposes the same marginal tax rate on the individuals that obtain the same amount of income regardless of whether they are in the non-cooperative and cooperative couples. Finally, the ratio of the distributions of non-cooperative and cooperative couples obtaining the same income is defined, respectively, by

$$\theta(w) \equiv \frac{\gamma(w)\pi(w)}{\gamma(w)\pi(w) + (1-\gamma(\widehat{w}(w)))\pi(\widehat{w}(w))}, \quad 1-\theta(w) = \frac{(1-\gamma(\widehat{w}(w)))\pi(\widehat{w}(w))}{\gamma(w)\pi(w) + (1-\gamma(\widehat{w}(w)))\pi(\widehat{w}(w))}.$$

Then the optimal marginal tax rates are given in the following proposition (Appendix G).

Proposition 3. *The optimal marginal tax rates in the economy with both the cooperative and non-cooperative couples are as follows. In the case with tagging,*

$$\frac{T_i'(w)}{1-T_i'(w)} = \theta(w)[A_i^n(w)B_i^n(w)C_i^n(w) + D_i^n(w)(1 + E_i^n(w))] \tag{45}$$
$$+ (1-\theta(w))A_i^c(\widehat{w}(w))B_i^c(\widehat{w}(w))C_i^c(\widehat{w}(w)),$$

for $i = a, b$, and $\widehat{w}(w)$ satisfying that $y_i^n(w) = y_i^c(\widehat{w}(w))$.

The marginal tax rate with tagging (45) is expressed by the weighted average of the tax formula for the non-cooperative couples, provided in Proposition 1, and the one for the cooperative couples, provided in Lemma 1, with the weight being the ratio of distributions for the non-cooperative couples with abilities $(w_a(w), w_b(w))$ and for the cooperative couples with abilities $(w_a(\widehat{w}(w)), w_b(\widehat{w}(w)))$ that obtain the same amount of income. Thus, it places the importance on the couple type (cooperative or non-cooperative) whose distribution is higher.

4.2 Housework-Specific Commodities

This section introduces housework-specific commodities for household production, which are substitutable for housework time. The household production provided by spouse i is modified as follows:

$$q_i = q_i(g_i + r_i(z_i)), \quad i = a, b, \tag{46}$$

where z_i is the amount of the housework-specific commodity that spouse i purchases. The function $r_i(\cdot)$ shows the substitutability between housework time and the housework-specific commodity, and it is assumed that $r_i' > 0 > r_i''$. As the existence of the housework-specific commodities makes the analysis extremely complex, for simplicity we assume a quasi-linear utility function, which is given by

$$u_i = x_i + h_i\left(1 - \frac{y_i}{w_i} - g_i\right) + q_i(g_i + r_i(z_i)) + q_j(g_j + r_j(z_j)), \quad i,j = a,b,\ i \neq j. \quad (47)$$

The budget constraint of each spouse is

$$x_i = y_i - T_i(y_i) - (1+t)z_i, \quad i = a,b, \quad (48)$$

where t is the tax/subsidy rate on z_i.

The first stage of the individual decisions process is modified as the Following. Each spouse non-cooperatively decides housework time and the amount of the housework-specific commodity.

See Appendix H for details on the subsequent analysis in this section. The first-order conditions with respect to y_i, g_i, and z_i of each spouse and the assortative mating assumption (12) imply that

$$y_i = y_i(w), \quad g_i = g_i(w,t,y_i(w)), \quad z_i = z_i(w,t,y_i(w)), \quad i = a,b. \quad (49)$$

Using this, (46) can be expressed as

$$q_i(w,t) = q_i(g_i(w,t,y_i(w)) + r_i(z_i(w,t,y_i(w)))), \quad i = a,b. \quad (50)$$

Let us define $g_{it} \equiv \partial g_i/\partial t$, $z_{it} \equiv \partial z_i/\partial t$, $\dot{z}_i \equiv \partial z_i/\partial w_i$, $q_{it} \equiv \partial q_i/\partial t$, $\varepsilon^i_{zT'} \equiv -\frac{1-T_i'}{z_i}\frac{\partial z_i}{\partial(1-T_i')}$, $\varepsilon^i_{r'y} \equiv \frac{y_i}{r_i'}\frac{\partial r_i'}{\partial y_i}$, $\varepsilon^i_{\dot{z}y} \equiv \frac{y_i}{\dot{z}_i}\frac{\partial \dot{z}_i}{\partial y_i}$. The details of these expressions are given in Appendix H. The following elasticities are used in the optimal commodity tax/subsidy expression:

$$\varepsilon_{qt}(w,t) \equiv -\frac{t \cdot \int_{\underline{w}}^{\overline{w}}(q_{at}(w,t) + q_{bt}(w,t))\pi(w)dw}{\int_{\underline{w}}^{\overline{w}}(q_a(w,t) + q_b(w,t))\pi(w)dw} > 0, \quad (51)$$

$$\varepsilon_{zt}(w,t) \equiv -\frac{t \cdot \int_{\underline{w}}^{\overline{w}}(z_{at}(w,t,y_a(w)) + z_{bt}(w,t,y_b(w)))\pi(w)dw}{\int_{\underline{w}}^{\overline{w}}(z_a(w,t,y_a(w)) + z_b(w,t,y_b(w)))\pi(w)dw} > 0. \quad (52)$$

ε_{qt} represents the tax-elasticity of total household public good $q_a + q_b$ and ε_{zt} the tax-elasticity of total demand on commodity $z_a + z_b$. Although each variable except y_i depends on t, we will omit t as an argument from now on to avoid notational verbosity. Additionally, let us define $\varrho_i(w) \equiv z_i(w)/l_i(w)$. Then, the optimal marginal tax rate is provided in the following proposition.

Proposition 4. *The optimal marginal tax rate in the presence of the housework-specific commodity is as follows. In the case with tagging,*

$$\frac{T_i'(w)}{1-T_i'(w)} = A_i^n(w)B_i^n(w)C_i^n(w) + D_i^n(w)(1+E_i^n(w)) + F_i^n(w), \tag{53}$$

where

$$\begin{aligned} F_i^n(w) \equiv & -\varrho_i(w) r_i'(z_i(w,y_i(w)))\frac{\varepsilon^i_{zT'}(w)}{\varepsilon^i_{lT'}(w)} - \frac{t}{1+t}\varrho_i(w) r_i'(z_i(w,y_i(w)))\frac{\varepsilon^i_{zT'}(w)}{\varepsilon^i_{lT'}(w)} \\ & - \int_w^{\overline{w}} \left(1 - \frac{\Phi_{u_j}(u_a(s),u_b(s))}{\lambda}\right)\pi(s)ds \frac{\varrho_i(w) r_i'(z_i(w,y_i(w)))}{\pi(w)w_i(w)} \\ & \cdot \frac{\varepsilon^i_{zT'}(w)}{\varepsilon^i_{lT'}(w)}\left(\varepsilon^i_{q'y}(w) + \varepsilon^i_{\dot{z}y}(w) - \varepsilon^i_{r'y}(w)\right) w_i'(w), \end{aligned} \tag{54}$$

for $i,j=a,b$ and $i \neq j$.

The optimal commodity tax/subsidy is given by

$$\frac{t}{1+t} = -\vartheta\frac{\varepsilon_{qt}}{\varepsilon_{zt}} - \Omega < 0, \tag{55}$$

where

$$\vartheta \equiv \frac{\int_{\underline{w}}^{\overline{w}}(q_a(w)+q_b(w))\pi(w)dw}{(1+t)\int_{\underline{w}}^{\overline{w}}(z_a(w,y_a(w))+z_b(w,y_b(w)))\pi(w)dw} > 0, \tag{56}$$

$$\begin{aligned} \Omega \equiv & -\left[\frac{1}{(1+t)\int_{\underline{w}}^{\overline{w}}(z_{at}(w,y_a(w))+z_{bt}(w,y_b(w)))\pi(w)dw}\right] \\ & \cdot \int_{\underline{w}}^{\overline{w}}\left\{\int_w^{\overline{w}}\left(1-\frac{\Phi_{u_a}(u_a(s),u_b(s))}{\lambda}\right)\pi(s)ds\right. \\ & \cdot \left[-h_a''\left(1-\frac{y_a(w)}{w_a(w)} - g_a(w,y_a(w))\right) g_{at}(w,y_a(w))\frac{y_a(w)}{(w_a(w))^2}\right] \\ & + \frac{\partial[q_b'(w)(\dot{g}_b(w,y_b(w)) + r_b'(z_b(w,y_b(w)))\dot{z}_b(w,y_b(w)))]}{\partial t} \\ & + \int_w^{\overline{w}}\left(1-\frac{\Phi_{u_b}(u_a(s),u_b(s))}{\lambda}\right)\pi(s)ds \\ & \cdot \left[-h_b''\left(1-\frac{y_b(w)}{w_b(w)} - g_b(w,y_b(w))\right) g_{bt}(w,y_b(w))\frac{y_b(w)}{(w_b(w))^2}\right. \\ & \left.\left. + \frac{\partial[q_a'(w)(\dot{g}_a(w,y_a(w)) + r_a'(z_a(w,y_a(w)))\dot{z}_a(w,y_a(w)))]}{\partial t}\right]\right\} dw > 0. \end{aligned} \tag{57}$$

The term F_i^n appears newly in the formula of the marginal tax rate in the existence of commodity z. The first term in F_i^n is a Pigouvian term allowing for the effects of the housework-specific commodity. The implications are similar to the expression D_i^n, relating to the housework time g_i. However, the direction of the impact on the optimal marginal tax rate is opposite in the housework time g_i and housework-specific commodity z_i. The second term in F_i^n relates to the revenue constraint: an increase in the subsidy rate to the commodities must raise the marginal income tax rate to compensate revenue. The final term in F_i^n can be interpreted in terms of relaxing the IC constraints of the partner through the variation of the household public good induced by the change in z_i. The intuition is analogous to that of the term $D_i^n E_i^n$ explained in Section 3.2.

The optimal commodity tax expression (55) comprises the Pigouvian term (first term) and labor-supply stimulation term (second term). The former shows that the subsidy improves the under-provision of the household public good. The latter describes the effect on labor supply through the change in housework time and is directly irrelevant to the under-provision of the household public good.

Optimal taxation on the commodities t always takes a negative sign at the optimum. The result contrasts with the Atkinson–Stiglitz theorem that the commodity taxes are not needed under optimal nonlinear income taxation. Saez (2002) shows that the optimal commodity tax rate is non-zero because of the effect of the heterogeneity in preferences across individuals on labor supply through budget constraints. Conversely, the nonzero commodity tax in our model stems from the effect of z_i on labor supply through adjustment of housework time g_i.

Under the Bentham criterion, $\Phi_{u_i} = 1$ for $i = a, b$, we have $\lambda = 1$.[12] Therefore, the optimal marginal income tax rate and optimal commodity tax/subsidy rate become

$$\begin{aligned} \frac{T_i'(w)}{1 - T_i'(w)} &= \beta_i(w)\frac{\varepsilon^i_{gT'}(w)}{\varepsilon^i_{lT'}(w)} - \varrho_i(w) r_i'(z_i(w, y_i(w)))\frac{\varepsilon^i_{zT'}(w)}{\varepsilon^i_{lT'}(w)} \\ &\quad - \left(\frac{t}{1+t}\right)\varrho_i(w) r_i'(z_i(w, y_i(w)))\frac{\varepsilon^i_{zT'}(w)}{\varepsilon^i_{lT'}(w)}, \quad i = a, b, \end{aligned} \tag{58}$$

and

$$\frac{t}{1+t} = -\frac{\varepsilon_{qt}}{\varepsilon_{zt}}\vartheta < 0. \tag{59}$$

The first term in the second equation in (58) captures a correction for the under-provision of the household public good through increased housework time, while the second term captures it through increased use of housework-specific commodities. The marginal tax rate tends to be positive as $\varepsilon^i_{gT'}$ increases, and tends to be negative if $\varepsilon^i_{zT'}$ increases. The larger value of $\varepsilon^i_{gT'}$ implies that a household public good intensively requires the time of spouses (e.g., childcare), while the larger value of $\varepsilon^i_{zT'}$ implies that it intensively requires housework-specific commodities (e.g., cleaning a room). The third term is the tax revenue effect, which shows that the income tax rate must increase (decrease) to compensate for changes in revenue caused by the subsidy adjustments to housework-specific commodities.

[12] As this section assumes the quasi-linear utility function, $\lambda = 1$ holds.

5 Conclusion

This study analyzed the optimal nonlinear income taxation for non-cooperative couples who under-provide the household public good. The household comprises two members engaging in non-cooperative behavior to contribute to the household public good. Considering the importance of time and effort for household public goods and the role of income taxation in encouraging housework time, supported by empirical evidence, we constructed a model with a substitute relationship between labor supply and housework time, where income tax can encourage housework. Incorporating non-cooperative household behavior into the Mirrleesian optimal income tax framework with the substitute relationship, we examined the structure of the optimal nonlinear income taxes.

The optimal marginal tax rate in this study was evaluated using the new Pigouvian form, in addition to the Mirrleesian ABC form provided in previous studies. The Pigouvian form suggests that income taxes should encourage more housework to improve the under-provision of the household public good. This consideration of income taxation is amplified by the effects of tax flexibility, which arise from the relaxation of the incentive compatibility (IC) constraint of a partner. Particularly, the marginal tax rate for the top is positive, which contrasts with the conventional result showing a zero marginal tax rate at the top.

Our quantitative analysis showed that the marginal tax rates for non-cooperative couples are higher than those of cooperative couples providing the household public good at an efficient level at all income levels. The difference arises because the Pigouvian consideration imposed on the income taxes for non-cooperative couples raises the marginal tax rates. This result suggests that the optimal marginal tax rates provided in previous studies, which do not account for non-cooperative behavior, may have been underestimated.

Recently, income tax reductions have been frequently implemented in developed countries. Particularly, reductions in the highest tax rate have been very pronounced over the last three decades. However, if household public goods are under-provided due to non-cooperative behavior, these policies may reduce social welfare level in terms of both efficiency and equity. For families with young children, income tax cuts may negatively impact child quality.

In addition, we note that some extensions remain for future research. First, if the utility of children is incorporated into our model, the optimal design of policies related to family size could be explored. Taking into account the empirical result that the time invested by both male and female spouses is crucial for human capital accumulation in children, this extension may demonstrate that the marginal tax rate becomes higher than that derived in our model to improve children's cognitive abilities. Second, we can allow for the choice of marriage and divorce driven by taxation. As the number of children increases with the number of married couples, the taxes should take the avoidance of divorce into account if children provide positive externalities to the society. The taxation makes marriage more attractive through the improvement of household public good provision.

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References

Ahmad, N., Koh, S. (2011). Incorporating estimates of household production of non-market services into international comparisons of material well-being. OECD Statistics Woking Papers 2011/7.

Apps, P., Rees, R. (1988). Taxation and the household. Journal of Public Economics, 35, 355–369.

Alesina, A., Ichino, A., Karabarbounis, L. (2011). Gender-based taxation and the division of family chores. American Economic Journal: Economic Policy, 3, 1–40.

Alves, C. B., da Costa, C. E., Lobel, F., Moreira, H. (2024). Intrahousehold inequality and the joint taxation of household earnings. Journal of Public Economics, 239.

Ashraf, N. (2009). Spousal control and intra-household decision making: An experimental study in the Philippines. American Economic Review, 99, 1245–1277.

Balestrino, A. Cigno, A., Pettini, A. (2002). Endogenous fertility and the design of family taxation. International Tax and Public Finance, 9, 175–193.

Bastani, S., Blomquist, S., Micheletto, L. (2020). Child care subsidies, quality, and optimal income taxation. American Economic Journal: Economic Policy, 12, 1–37.

Becker, G. S. (1965). A theory of the allocation of time. The Economic Journal, 75, 493–517.

Boskin, M. J., Sheshinski, E. (1983). Optimal tax treatment of the family: Married couples. Journal of Public Economics, 20, 281–297.

Bierbrauer, F.J., Boyer, P. C., Peichl, A., Weishaar, D. (2023). The taxation of couples. CESifo Working Paper Series 10414.

Carbone, J., Cahn, N. (2014). Marriage markets: how inequality is remaking the American family. Oxford and New York: Oxford University Press.

Chetty, R. (2006). A new method of estimating risk aversion. American Economic Review, 96, 1821–834.

Chetty, R., Guren, A., Manoli, D., Weber, A. (2011). Are micro and macro labor supply elasticities consistent? A review of evidence on the intensive and extensive margins. American Economic Review, 10, 471–475.

Chiappori, P. (1988). Rational household labor supply. Econometrica, 56, 63–90.

Cigno, A. (2012). Marriage as a commitment device. Review of Economics of the Household, 10, 193–213.

Cremer, H., Gahvari, F., Lozachmeur, J.-M. (2010). Tagging and income taxation: Theory and an application. American Economic Journal: Economic Policy, 2, 31–50.

Cremer, H., Lozachmeur, J.-M., Maldonado, D., Roeder, K. (2016). Household bargaining and the design of couples' income taxation. European Economic Review, 89, 454–470.

Cochard, F., Couprie, H., Hopfensitz, A. (2016). Do spouses cooperate? An experimental investigation. Review of Economics of the Household, 14, 1–26.

Cornes, R., Itaya, J., Tanaka, A. (2012). Private provision of public goods between families. Journal of Population Economics. 25, 1451–1480.

Del Boca, D., Flinn, C. (2012). Endogenous household interaction. Journal of Econometrics, 166, 49–65.

Del Boca, D., Flinn, C., Wiswall, M. (2014). Household choices and child development. Review of Economic Studies, 81, 137–185.

Diamond, P., A. (1988). Optimal income taxation: An example with a U-shaped pattern of optimal marginal tax rates, The American Economic Review, 88, 83–95.

Doepke, M., Kindermann, F. (2019). Bargaining over babies: Theory, evidence, and policy implications. American Economic Review, 109, 3264–306.

Doepke, M., Tertilt, T. (2019). Does female empowerment promote economic development? Journal of Economic Growth, 24, 309–343.

Eika, L., Mogstad, M., Zafar, B. (2019). Educational assortative mating and household income inequality. Journal of Political Economy, 127, 2795–2835.

Frankel, A. (2014). Taxation of couples under assortative mating. American Economic Journal: Economic Policy, 6, 155–177.

Gobbi, P. E. (2018). Childcare and commitment within households. Journal of Economic Theory, 176, 503–551.

Golosov, M., Krasikov, I. (2023). The optimal taxation of couples, NBER Working Paper 31140.

Gayle, G., Shephard, A. (2019). Optimal taxation, marriage, home production, and family labor supply. Econometrica, 87, 291–326.

Heath, R., Tan, X. (2020). Intrahousehold bargaining, female autonomy, and labor supply: Theory and evidence from India. Journal of the European Economic Association, 18, 1928–1968.

Ho, C., Pavoni, N. (2020). Efficient child care subsidies. American Economic Review, 110, 162–199.

Itaya, j., de Meza, D., Myles, G. (2002). Income distribution, taxation, and the private provision of public goods. Journal of Public Economic Theory, 4, 273–297.

Kabátek, J., van Soest, V., Stancanelli, E. (2014). Income taxation, labour supply and housework: A discrete choice model. Labour Economics, 27, 30–43.

Kenney, C. T. (2006). The power of the purse: Allocative systems and inequality in couple households. Gender and Society, 20, 354–381.

Kleven, H. J., Kreiner, C. T., Saez, E. (2009). The optimal income taxation of couples. Econometrica, 77, 537–560.

Konrad, K. A., Lommerud, K. E. (1995). Family policy with non-cooperative families. Scandinavian Journal of Economics, 97, 581–601.

Kurnaz, M. (2021). Optimal taxation of families: Mirrlees meets Becker. The Economic Journal, 131, 2984–3011.

Lauer, S. R., Yodanis, C. (2014). Money management, gender and households. In J. Treas, J. Scott, M. Richards (Eds.), The Wiley Blackwell Companion of the Sociology of Families, 344–360. Hoboken, NJ: Wiley.

Lechene, V., Preston, I. (2011). Noncooperative household demand. Journal of Economic Theory, 146, 504–527.

Lundborg, P, Nilsson, A, Rooth DO. (2014). Parental education and offspring outcomes: Evidence from the Swedish compulsory schooling reform., American Economic Journal: Applied 6, 253–278.

Lundberg, S., Pollak, R., A. (1995). Separate spheres bargaining and the marriage market. Journal of Political Economy, 101, 988–1010.

Mankiw, N. G., Weinzierl, H., Yagan, D. (2009). Optimal taxation in theory and practice, Journal of Economic Perspectives, 23, 147–174.

Mirrlees, J. A. (1971). An exploration in the theory of optimal income taxation. Review of Economic Studies, 38, 175–208.

Mirrlees, J. A. (1976). Optimal tax theory: A synthesis. Journal of Public Economics, 6, 327–358.

Meier, V., Rainer, H. (2015). Pigou meets Ramsey: Gender-based taxation with non-cooperative couples. European Economic Review, 77, 28–46.

Obara, T. (2019). Optimal human capital policies under the endogenous choice of educational types. Journal of Public Economic Theory, 21, 512–536.

Obara, T., Ogawa, Y. (2024). Optimal taxation in an endogenous fertility model with non-cooperative behavior. Review of Economics of the Household, forthcoming.

Pahl, J. (2008). Family finances, individualisation, spending patterns and access to credit. Journal of Socio-Economics, 37, 577–591.

Saez, E. (2001). Using elasticities to derive optimal income tax rates. Review of Economic Studies, 68, 205–229.

Saez, E. (2002). Optimal income transfer programs: Intensive versus extensive labor supply responses. Quarterly Journal of Economics, 117, 1039–1073.

Stiglitz, J. E. (1982). Self-selection and pareto efficient taxation. Journal of Public Economics, 17, 213–240.

Samuelson, P. (1956). Social indifference curves. Quarterly Journal of Economics, 70, 1–22.

Schroyen, F. (2003). Redistributive taxation and the household: The case of individual filings. Journal of Public Economics, 87, 2527–2547.

Appendix A: The Derivation of the IC Condition

The incentive-compatibility constraints requiring that each spouse in a couple of type $(w_a(w), w_b(w))$ choose the consumption-income bundle intended for it, not choosing the bundle for $(w_a(w^*), w_b(w^*))$, are as follows.

$$\begin{aligned} u_i(w) \quad \geq \quad & v_i(x_i(w^*)) + h_i\left(1 - \frac{y_i(w^*)}{w_i(w)} - g_i(w, y_i(w^*))\right) \\ & + q_i(g_i(w, y_i(w^*))) + q_j(g_j(w, y_j(w^*))), \quad i, j = a, b,\ i \neq j, \end{aligned} \tag{60}$$

for any w and w^*. As the government knows type $w_a(w)$ is a couple with type $w_b(w)$, type $w_a(w)$ cannot declare to be a couple with type $w_b(w^*)$ for $w \neq w^*$. Thus, we consider the IC constraints choosing the bundle for $(w_a(w^*), w_b(w^*))$ when each spouse mimics.

Here, we rewrite (60) as a minimization problem. For type $(w_a(w^*), w_b(w^*))$, the counterpart of (60) is

$$\begin{aligned} u_i(w^*) \quad \geq \quad & v_i(x_i(w)) + h_i\left(1 - \frac{y_i(w)}{w_i(w^*)} - g_i(w^*, y_i(w))\right) \\ & + q_i(g_i(w^*, y_i(w))) + q_j(g_j(w^*, y_j(w))), \quad i, j = a, b,\ i \neq j. \end{aligned} \tag{61}$$

Thus, we have

$$\begin{aligned} 0 = & u_i(w) - v_i(x_i(w)) - h_i\left(1 - \frac{y_i(w)}{w} - g_i(w, y_i(w))\right) \\ & + q_i(g_i(w, y_i(w))) + q_j(g_j(w, y_j(w))) \\ \leq & u_i(w^*) - v_i(x_i(w)) - h_i\left(1 - \frac{y_i(w)}{w_i(w^*)} - g_i(w^*, y_i(w))\right) \\ & - q_i(g_i(w^*, y_i(w))) - q_j(g_j(w^*, y_j(w))), \quad i, j = a, b,\ i \neq j. \end{aligned} \tag{62}$$

This implies that w^* minimizes the right-hand side of (62) at $w = w^*$. Evaluating the first-order condition at $w = w^*$, we obtain

$$\begin{aligned} \dot{u}_i(w) = & h_i'\left(1 - \frac{y_i(w)}{w_i(w)} - g_i(w, y_i(w))\right)\frac{y_i(w)w_i'(w)}{(w_i(w))^2} \\ & + q_j'(g_j(w, y_j(w)))\dot{g}_j(w, y_j(w))w_j'(w), \quad i, j = a, b,\ i \neq j, \end{aligned} \tag{63}$$

for any w. This is the set of first-order incentive compatibility (FOIC) conditions, which correspond to (19). The first-order approach uses only these FOIC conditions.

Appendix B: The Derivations of the Elasticities

From (9), we obtain

$$\left.\frac{\partial l_i}{\partial(1 - T_i')}\right|_{u_i = \text{constant}} = -\frac{v_i'}{(1 - T_i')^2 w_i v_i'' + \left(\frac{1}{w_i} + g_{iy_i}\right) h_i''} > 0, \quad i = a, b. \tag{64}$$

Using this and (9) again, we obtain (20). Introducing the virtual income yields

$$\eta_l^i = \varepsilon_{lT'}^{ui} - \varepsilon_{lT'}^{ci} = -\frac{(1-T_i')^2 w_i^2 v_i''}{\left(1-T_i'\right)^2 w_i^2 v_i'' + \left(1+w_i g_{iy_i}\right) h_i''} < 0, \quad i = a, b. \tag{65}$$

From this and (20), we obtain (21).

Noting that $w_i = dy_i/dl_i$ and (11), we obtain

$$\left.\frac{\partial g_i}{\partial(1-T_i')}\right|_{u_i=\text{constant}} = w_i g_{iy_i} \left.\frac{\partial l_i}{\partial(1-T_i')}\right|_{\overline{u}_i=\text{constant}} < 0, \quad i = a, b.$$

This fact and (64) lead to (22). (23) is derived from (2), (8), and (11), and (24) is derived from (7).

Appendix C: The Proof of Lemma 1

In our framework, because the government cannot observe housework time g_i and private consumption x_i, it indirectly manipulates unobservable housework time and private consumption by adjusting the observable income y_i and household's disposable income I. To express g_i as a function of y_i and x_i as a function of I, we disaggregate individual optimization process into two stages. First, the household determines the spouses' labor supply and the household's disposable income. Next, it decides the spouses' time devoted to household production and private consumption.

The household budget constraint in the second stage is

$$x_a + x_b = I. \tag{66}$$

The household maximizes (25) with respect to x_i and g_i subject to (66). The corresponding Lagrangian to this maximization problem is

$$\begin{aligned} L^{c2} \equiv\ & v_a(x_a) + v_b(x_b) + h_a\left(1 - \frac{y_a}{w_a} - g_a\right) + h_b\left(1 - \frac{y_b}{w_b} - g_b\right) \\ & + 2q_a(g_a) + 2q_b(g_b) + \psi \cdot (I - x_a - x_b), \end{aligned} \tag{67}$$

where ψ is the multiplier associated with the binding budget constraint. The FOCs with respect to x_i, ψ, and g_i are

$$x_i : v_i'(x_i) - \psi = 0, \quad i = a, b, \tag{68}$$

$$\psi : I - x_a - x_b = 0, \tag{69}$$

$$g_i : -h_i'\left(1 - \frac{y_i}{w_i} - g_i\right) + 2q_i'(g_i) = 0, \quad i = a, b. \tag{70}$$

The final term $2q_i'$ in (70), which differs from that of the non-cooperative couple, implies that the household allows for the effect of the household public good on both spouses. From (68) and (69), we have

$$x_i = x_i(I), \quad i = a, b. \tag{71}$$

From (70),

$$g_i = g_i(w_i, y_i), \quad i = a, b. \tag{72}$$

Next we consider the optimization problem of the household in the second stage. Substituting (71) and (72) into (25) yields

$$\begin{aligned} u &= v_a(x_a(I)) + v_b(x_b(I)) + h_a\left(1 - \frac{y_a}{w_a} - g_a(w_a, y_a)\right) \\ &\quad + h_b\left(1 - \frac{y_b}{w_b} - g_b(w_b, y_b)\right) + 2q_a(g_a(w_a, y_a)) + 2q_b(g_b(w_b, y_b)). \end{aligned} \tag{73}$$

The household budget constraint in the first stage is given by

$$I = y_a - T_a(y_a) + y_b - T_b(y_b). \tag{74}$$

The corresponding Lagrangian to this maximization problem is

$$\begin{aligned} L^{c1} &\equiv v_a(x_a(I)) + v_b(x_b(I)) + h_a\left(1 - \frac{y_a}{w_a} - g_a(w_a, y_a)\right) \\ &\quad + h_b\left(1 - \frac{y_b}{w_b} - g_b(w_b, y_b)\right) + 2q_a(g_a(w_a, y_a)) \\ &\quad + 2q_b(g_b(w_b, y_b)) + \varpi \cdot (y_a - T_a(y_a) + y_b - T_b(y_b) - I), \end{aligned}$$

where ϖ is the multiplier associated with the binding budget constraint. The FOCs with respect to y_i, ϖ, and I are

$$y_i : -h_i'\left(1 - \frac{y_i}{w_i} - g_i(w_i, y_i)\right)\frac{1}{w_i} + \varpi(1 - T_i'(y_i)) = 0, \quad i = a, b, \tag{75}$$

$$\varpi : y_a - T_a(y_a) + y_b - T_b(y_b) - I = 0, \tag{76}$$

$$I : v_a'(x_a(I))x_a'(I) + v_b'(x_b(I))x_b'(I) - \varpi = 0, \tag{77}$$

where (70) is used to derive (75). From (66) and (71), the following holds

$$x_a'(I) + x_b'(I) = 1. \tag{78}$$

Applying this and (68) to (77), we obtain

$$v_a'(x_a(I)) = v_b'(x_b(I)) = \varpi. \tag{79}$$

Using (79), (75) becomes

$$-h_i'\left(1 - \frac{y_i}{w_i} - g_i(w_i, y_i)\right)\frac{1}{w_i} + v_i'(x_i(I))(1 - T_i'(y_i)) = 0, \quad i = a, b. \tag{80}$$

From (74) and (80) we have

$$y_i = y_i(w_i, w_j), \quad i, j = a, b, \ i \neq j. \tag{81}$$

From the assumption of the assortative mating of couples, we express

$$y_i(w) = y_i(w_i(w), w_j(w)), \quad i, j = a, b, \ i \neq j. \tag{82}$$

From the assumption of assortative mating and (72),

$$g_i(w, y_i(w)) = g_i(w_i(w), y_i(w_i(w), w_j(w))), \quad i, j = a, b, \ i \neq j. \tag{83}$$

We discuss incentive-compatibility constraints of cooperative couples. Using (82) and (83), (25) and (74) lead to

$$\begin{aligned} u(w) \quad = \quad & v_a(x_a(I(w))) + v_b(x_b(I(w))) + h_a\left(1 - \frac{y_a(w)}{w_a(w)} - g_a(w, y_a(w))\right) \\ & + h_b\left(1 - \frac{y_b(w)}{w_b(w)} - g_b(w, y_b(w))\right) \\ & + 2q_a(g_a(w, y(w))) + 2q_b(g_b(w, y_b(w))), \end{aligned} \tag{84}$$

where

$$I(w) = y_a(w) - T_a(y_a(w)) + y_b(w) - T_b(y_b(w)). \tag{85}$$

Using the context of Appendix A, (70), and (78)–(80), we obtain

$$\begin{aligned} \dot{u} \quad = \quad & h'_a\left(1 - \frac{y_a(w)}{w_a(w)} - g_a(w, y_a(w))\right)\frac{y_a(w)w'_a(w)}{(w_a(w))^2} \\ & + h'_b\left(1 - \frac{y_b(w)}{w_b(w)} - g_b(w, y_b(w))\right)\frac{y_b(w)w'_b(w)}{(w_b(w))^2}. \end{aligned} \tag{86}$$

The welfare criterion that sums over all individuals a transformation $\Phi(u)$ of individual utility with $\Phi'(\equiv d\Phi/du) > 0$ and $\Phi''(\equiv d\Phi'/du) \leq 0$ (hence the government has a non-negative aversion to inequality) and Φ is independent of w, that is,

$$\int_{\underline{w}}^{\overline{w}} \Phi(u(w))\pi(w)dw. \tag{87}$$

From (73), we have the following implicit function:

$$\widetilde{I}(w) = \widetilde{I}(u(w), y_a(w), y_b(w)), \tag{88}$$

which satisfies

$$\widetilde{I}_{y_i}\left(\equiv \frac{\partial \widetilde{I}}{\partial y_i}\right) = \frac{h'_i}{w_i v'_i}, \quad \widetilde{I}_u\left(\equiv \frac{\partial \widetilde{I}}{\partial u}\right) = \frac{1}{v'_i}, \quad i = a, b. \tag{89}$$

The government budget constraint is given by

$$\int_{\underline{w}}^{\overline{w}} (y_a(w) + y_b(w) - \widetilde{I}(u(w), y_a(w), y_b(w)))\pi(w)dw \geq R. \tag{90}$$

The problem for the government is to choose $y_a(w)$, $y_b(w)$, and $u(w)$ to maximize the welfare function (87) subject to the budget constraint (90) and the IC constraint (86). The corresponding Lagrangian to the maximization problem is

$$\begin{aligned} L^{cg} \equiv & \int_{\underline{w}}^{\overline{w}} \Phi(u(w))\pi(w)dw + \lambda \cdot \int_{\underline{w}}^{\overline{w}} (y_a(w) + y_b(w) - \widetilde{I}(u(w), y_a(w), y_b(w)) \\ & - R)\pi(w)dw + \int_{\underline{w}}^{\overline{w}} \varsigma(w) \left(h_a' \left(1 - \frac{y_a(w)}{w_a(w)} - g_a(w, y_a(w)) \right) \frac{y_a(w)w_a'(w)}{(w_a(w))^2} \right. \\ & \left. + h_b' \left(1 - \frac{y_b(w)}{w_b(w)} - g_b(w, y_b(w)) \right) \frac{y_b(w)w_b'(w)}{(w_b(w))^2} - \dot{u}(w) \right) dw, \end{aligned} \tag{91}$$

where λ is the multiplier associated with the binding budget constraint and $\varsigma(w)$ is the multiplier associated with the IC constraint.

Note that the following holds:

$$\int_{\underline{w}}^{\overline{w}} \varsigma(w)\dot{u}(w)dw = \varsigma(\overline{w})u(\overline{w}) - \varsigma(\underline{w})u(\underline{w}) - \int_{\underline{w}}^{\overline{w}} \dot{\varsigma}(w)u(w)dw. \tag{92}$$

Using this, (91) can be rewritten as

$$\begin{aligned} L^{cg} \equiv & \int_{\underline{w}}^{\overline{w}} \Phi(u(w))\pi(w)dw - \varsigma(\overline{w})u(\overline{w}) + \varsigma(\underline{w})u(\underline{w}) \\ & + \lambda \int_{\underline{w}}^{\overline{w}} (y_a(w) + y_b(w) - \widetilde{I}(u(w), y_a(w), y_b(w)) - R)\pi(w)dw \\ & + \int_{\underline{w}}^{\overline{w}} \varsigma(w) \left(h_a' \left(1 - \frac{y_a(w)}{w_a(w)} - g_a(w, y_a(w)) \right) \frac{y_a(w)w_a'(w)}{(w_a(w))^2} \right. \\ & \left. + h_b' \left(1 - \frac{y_b(w)}{w_b(w)} - g_b(w, y_b(w)) \right) \frac{y_b(w)w_b'(w)}{(w_b(w))^2} \right) dw \\ & + \int_{\underline{w}}^{\overline{w}} \dot{\varsigma}(w)u(w)dw. \end{aligned} \tag{93}$$

Using (89), the FOCs are

$$\begin{aligned} y_i(w) : 0 = & \lambda\pi(w) \left(1 - \frac{h_i' \left(1 - \frac{y_i(w)}{w_i(w)} - g_i(w, y_i(w)) \right)}{v_i'(x_i(w))w_i(w)} \right) \\ & + \varsigma(w) \left[\frac{h_i' \left(1 - \frac{y_i(w)}{w_i(w)} - g_i(w, y_i(w)) \right)}{(w_i(w))^2} \right. \\ & - \frac{h_i'' \left(1 - \frac{y_i(w)}{w_i(w)} - g_i(w, y_i(w)) \right) y_i(w)}{(w_i(w))^2} \\ & \left. \cdot \left(\frac{1}{w_i(w)} + g_{iy_i}(w, y_i(w)) \right) \right] w_i'(w), \quad i = a, b, \end{aligned} \tag{94}$$

$$u(w): 0 = \left(\Phi'(u(w)) - \frac{\lambda}{v_i'(x_i(w))}\right)\pi(w) + \dot{\varsigma}(w), \tag{95}$$

$$u(\underline{w}): 0 = \varsigma(\underline{w}), \tag{96}$$

$$u(\overline{w}): 0 = \varsigma(\overline{w}). \tag{97}$$

Integrating $\dot{\varsigma}(w)$ and making use of (97) yields

$$\int_w^{\overline{w}} \dot{\varsigma}(s)ds = \varsigma(\overline{w}) - \varsigma(w) = -\varsigma(w).$$

Integrating (95) yields

$$0 = \int_w^{\overline{w}} \left(\Phi'(u(s)) - \frac{\lambda}{v_i'(x_i(s))}\right)\pi(s)ds + \int_w^{\overline{w}} \dot{\varsigma}(s)ds.$$

These two equations yield

$$-\frac{\varsigma(w)}{\lambda} = \int_w^{\overline{w}} \left(\frac{1}{v_i'(x_i(s))} - \frac{\Phi'(u(s))}{\lambda}\right)\pi(s)ds. \tag{98}$$

Using (80) and (98), (94) leads to

$$\begin{aligned}\frac{T_i'(w)}{1-T_i'(w)} = \int_w^{\overline{w}} \left(\frac{1}{v_i'(x_i(s))} - \frac{\Phi'(u(s))}{\lambda}\right)\pi(s)ds \frac{v_i'(x_i(w))w_i'(w)}{w_i(w)\pi(w)} \\ \cdot \left[1 - \frac{h_i''(\cdot)y_i(w)}{h_i'(\cdot)}\left(\frac{1}{w_i(w)} + g_{iy_i}(\cdot)\right)\right], \quad i = a, b.\end{aligned} \tag{99}$$

Applying (117) in Appendix D and the definitions of A_i^c, B_i^c, and C_i^c to (99) yields Lemma 1.[13]

Appendix D: The Proof of Proposition 1

Before proving Proposition 1, we provide the implicit function of x_i with respect to $u_i(w)$, $y_a(w)$, and $y_b(w)$, which is used in solving the optimal control problem. From (16), the implicit function of x_i is given by

$$x_i(w) = X_i(u_i(w), y_i(w), y_j(w)), \quad i, j = a, b, \ i \neq j, \tag{100}$$

which satisfies[14]

$$\frac{\partial X_i(u_i(w), y_i(w), y_j(w))}{\partial y_i(w)} = \frac{h_i'}{v_i' w_i}, \quad i, j = a, b, \ i \neq j, \tag{101}$$

[13]Note that the definitions of $\varepsilon_{lT'}^{ui}$ and $\varepsilon_{lT'}^{ci}$ is the same between the non-cooperative and cooperative cases.

[14]We used (5) to derive (101).

$$\frac{\partial X_i(u_i(w), y_i(w), y_j(w))}{\partial y_j(w)} = -\frac{q_j' g_{jy_j}}{v_i'}, \quad i, j = a, b, \ i \neq j, \tag{102}$$

$$\frac{\partial X_i(u_i(w), y_i(w), y_j(w))}{\partial u_i(w))} = \frac{1}{v_i'}, \quad i, j = a, b, \ i \neq j. \tag{103}$$

Using (100), the government budget constraint takes the form:

$$\int_{\underline{w}}^{\overline{w}} (y_a(w) - X_a(u_a(w), y_a(w), y_b(w)) + y_b(w) - X_b(u_b(w), y_b(w), y_a(w)))\pi(w)dw \geq R. \tag{104}$$

The problem for the government is to choose $y_a(w)$, $y_b(w)$, $u_a(w)$, and $u_b(w)$ maximize the social welfare function (17) subject to the budget constraint (104) and IC constraint (19). The corresponding Lagrangian to the maximization problem is

$$\begin{aligned} L^n \equiv & \int_{\underline{w}}^{\overline{w}} \Phi(u_a(w), u_b(w))\pi(w)dw + \lambda \int_{\underline{w}}^{\overline{w}} (y_a(w) - X_a(u_a(w), y_a(w), y_b(w)) \\ & + y_b(w) - X_b(u_b(w), y_b(w), y_a(w)) - R)\, \pi(w)dw \\ & + \int_{\underline{w}}^{\overline{w}} \varsigma_a(w) \left(h_a' \left(1 - \frac{y_a(w)}{w_a(w)} - g_a(w, y_a(w)) \right) \frac{y_a(w) w_a'(w)}{(w_a(w))^2} \right. \\ & \left. + q_b'(g_b(w, y_b(w)))\dot{g}_b(w, y_b(w))w_b'(w) - \dot{u_a}(w) \right) dw \\ & + \int_{\underline{w}}^{\overline{w}} \varsigma_b(w) \left(h_b' \left(1 - \frac{y_b(w)}{w_b(w)} - g_b(w, y_b(w)) \right) \frac{y_b(w) w_b'(w)}{(w_b(w))^2} \right. \\ & \left. + q_a'(g_a(w, y_a(w)))\dot{g}_a(w, y_a(w))w_a'(w) - \dot{u_b}(w) \right) dw, \end{aligned} \tag{105}$$

where λ is the multiplier associated with the binding budget constraint and $\varsigma_i(w)$ is the multiplier associated with the IC constraint.

Note that the following holds:

$$\int_{\underline{w}}^{\overline{w}} \varsigma_i(w)\dot{u}_i(w)dw = \varsigma_i(\overline{w})u_i(\overline{w}) - \varsigma_i(\underline{w})u_i(\underline{w}) - \int_{\underline{w}}^{\overline{w}} \dot{\varsigma}_i(w)u_i(w)dw, \quad i = a, b. \tag{106}$$

Using (106), (105) becomes

$$
\begin{aligned}
L^n \equiv & \int_{\underline{w}}^{\overline{w}} \Phi(u_a(w), u_b(w))\pi(w)dw + \lambda \int_{\underline{w}}^{\overline{w}} (y_a(w) - X_a(u_a(w), y_a(w), y_b(w)) \\
& + y_b(w) - X_b(u_b(w), y_b(w), y_a(w)) - R)\pi(w)dw \\
& - \varsigma_a(\overline{w})u_a(\overline{w}) + \varsigma_a(\underline{w})u_a(\underline{w}) - \varsigma_b(\overline{w})u_b(\overline{w}) + \varsigma_b(\underline{w})u_b(\underline{w}) \\
& + \int_{\underline{w}}^{\overline{w}} \varsigma_a(w) \left(h_a' \left(1 - \frac{y_a(w)}{w_a(w)} - g_a(w, y_a(w)) \right) \frac{y_a(w)w_a'(w)}{(w_a(w))^2} \right. \\
& \left. + q_b'(g_b(w, y_b(w)))\dot{g}_b(w, y_b(w))w_b'(w) \right) dw + \int_{\underline{w}}^{\overline{w}} \dot{\varsigma}_a(w)u_a(w)dw \\
& + \int_{\underline{w}}^{\overline{w}} \varsigma_b(w) \left(h_b' \left(1 - \frac{y_b(w)}{w_b(w)} - g_b(w, y_b(w)) \right) \frac{y_b(w)w_b'(w)}{(w_b(w))^2} \right. \\
& \left. + q_a'(g_a(w, y_a(w)))\dot{g}_a(w, y_a(w))w_a'(w) \right) dw + \int_{\underline{w}}^{\overline{w}} \dot{\varsigma}_b(w)u_b(w)dw.
\end{aligned} \tag{107}
$$

Allowing for (100)–(103), the necessary conditions (assuming an interior solution) are

$$
\begin{aligned}
y_i(w) : 0 = \lambda\pi(w) & \left(1 - \frac{h_i'\left(1 - \frac{y_i(w)}{w_i(w)} - g_i(w, y_i(w))\right)}{v_i'(x_i(w))w_i(w)} \right. \\
& \left. + \frac{q_i'(g_i(w, y_i(w)))g_{iy_i}(w, y_i(w))}{v_j'(x_j(w))} \right) + \varsigma_i(w) \left[\frac{h_i'\left(1 - \frac{y_i(w)}{w_i(w)} - g_i(w, y_i(w))\right)}{(w_i(w))^2} \right. \\
& \left. - \frac{h_i''\left(1 - \frac{y_i(w)}{w_i(w)} - g_i(w, y_i(w))\right) y_i(w)}{(w_i(w))^2} \left(\frac{1}{w_i(w)} + g_{iy_i}(w, y_i(w)) \right) \right] w_i'(w) \\
& + \varsigma_j(w) \left(q_i''(g_i(w, y_i(w)))\dot{g}_i(w, y_i(w))g_{iy_i}(w, y_i(w)) \right. \\
& \left. + q_i'(g_i(w, y_i(w)))\frac{\partial \dot{g}_i(w, y_i(w))}{\partial y_i(w)} \right) w_i'(w), \quad i, j = a, b, \ i \neq j,
\end{aligned} \tag{108}
$$

$$
u_i(w) : 0 = \left(\Phi_{u_i}(u_a(w), u_b(w)) - \frac{\lambda}{v_i'(x_i(w))} \right) \pi(w) + \dot{\varsigma}_i(w), \quad i = a, b, \tag{109}
$$

$$
u_i(\underline{w}) : 0 = \varsigma_i(\underline{w}), \quad i = a, b, \tag{110}
$$

$$
u_i(\overline{w}) : 0 = \varsigma_i(\overline{w}), \quad i = a, b. \tag{111}
$$

Integrating $\dot{\varsigma}_i(w)$ and making use of (111) yields

$$
\int_{w}^{\overline{w}} \dot{\varsigma}_i(s)ds = \varsigma_i(\overline{w}) - \varsigma_i(w) = -\varsigma_i(w), \quad i = a, b. \tag{112}
$$

Integrating (109) yields

$$0 = \int_{\underline{w}}^{\overline{w}} \left(\Phi_{u_i}(u_a(s), u_b(s)) - \frac{\lambda}{v_i'(x_i(s))} \right) \pi(s) ds + \int_{\underline{w}}^{\overline{w}} \dot{\varsigma}_i(s) ds, \quad i = a, b. \tag{113}$$

(112) and (113) yield

$$\frac{\varsigma_i(w)}{\lambda} = -\int_{w}^{\overline{w}} \left(\frac{1}{v_i'(x_i(s))} - \frac{\Phi_{u_i}(u_a(s), u_b(s))}{\lambda} \right) \pi(s) ds, \quad i = a, b. \tag{114}$$

Using (5), (9), and (114), (108) leads to

$$\begin{aligned} \frac{T_i'(w)}{1 - T_i'(w)} = & \int_{w}^{\overline{w}} \left(\frac{1}{v_i'(x_i(s))} - \frac{\Phi_{u_i}(u_a(s), u_b(s))}{\lambda} \right) \pi(s) ds \frac{w_i'(w) v_i'(x_i(w))}{\pi(w) w_i(w)} \\ & \cdot \left[1 - \frac{h_i''(\cdot) y_i(w)}{h_i'(\cdot)} \left(\frac{1}{w_i(w)} + g_{iy_i}(\cdot) \right) \right] \\ & + \int_{w}^{\overline{w}} \left(\frac{1}{v_j'(x_j(s))} - \frac{\Phi_{u_j}(u_a(s), u_b(s))}{\lambda} \right) \pi(s) ds \frac{w_i'(w)}{(1 - T_i'(w)) \pi(w)} \\ & \cdot \left(q_i''(\cdot) \dot{g}_i(\cdot) g_{iy_i}(\cdot) + q_i'(\cdot) \frac{\partial \dot{g}_i(\cdot)}{\partial y_i(w)} \right) - \frac{w_i(w) g_{iy_i}(\cdot) v_i'(x_i(w))}{v_j'(x_j(w))}, \\ & \qquad i, j = a, b, \ i \neq j. \end{aligned} \tag{115}$$

From (5), (7), (8), and (9), we obtain

$$\begin{aligned} & \frac{1}{1 - T_i'(w)} \left(q_i''(\cdot) \dot{g}_i(\cdot) g_{iy_i}(\cdot) + q_i'(\cdot) \frac{\partial \dot{g}_i(\cdot)}{\partial y_i(w)} \right) \\ & = v_i'(x_i(w)) g_{iy_i}(\cdot) \left(\frac{y_i(w)}{q_i'(\cdot)} \frac{\partial q_i'(\cdot)}{\partial y_i(w)} + \frac{y_i(w)}{\dot{g}_i(\cdot)} \frac{\partial \dot{g}_i(\cdot)}{\partial y_i(w)} \right), \quad i = a, b. \end{aligned} \tag{116}$$

From (20) and (21), after some manipulations we have

$$1 - \frac{h_i'' y_i}{h_i'} \left(\frac{1}{w_i} + g_{iy_i} \right) = \frac{1 + \varepsilon_{lT'}^{ui}}{\varepsilon_{lT'}^{ci}}, \quad i = a, b. \tag{117}$$

Using (20), (22), and the definition of β_i, we have

$$-w_i g_{iy_i} = \beta_i \frac{\varepsilon_{gT'}^{ci}}{\varepsilon_{lT'}^{ci}}, \quad i = a, b. \tag{118}$$

Applying (116)–(118) to (115) and making use of the definitions of A_i^n, B_i^n, C_i^n, D_i^n, and E_i^n yields (30).

Appendix E: The Proof of Corollary 1

The utility functions of the spouses are

$$u_i = v_i\left(y_i - T_i(y_i)\right) + h_i\left(1 - \frac{y_i}{w_i} - g_i\right) + q_b(g_b), \quad i = a, b. \tag{119}$$

The first-order condition with respect to g_b takes the same form as (5) for $i = b$ and hence (6)–(8) holds for $i = b$.

Substituting (6) for $i = b$ into (119) and maximizing the resulting utility function with respect to y_i yields[15]

$$y_a : 0 = v_a'\left(y_a - T_a(y_a)\right)\left(1 - T_a'(y_a)\right) - h_a'\left(1 - \frac{y_a}{w_a}\right)\frac{1}{w_a}, \tag{120}$$

$$y_b : 0 = v_b'\left(y_b - T_b(y_b)\right)\left(1 - T_b'(y_b)\right) - h_b'\left(1 - \frac{y_b}{w_b} - g_b(w_b, y_b)\right)\frac{1}{w_b}, \tag{121}$$

which implies that

$$y_i = y_i(w_i), \quad i = a, b. \tag{122}$$

From this and $g_b = g_b(w_b, y_b)$,

$$g_b = g_b(w_b, y_b(w_b)). \tag{123}$$

Allowing for (12), we can express the allocations of the couple as the functions of w:

$$y_i = y_i(w), \quad i = a, b, \tag{124}$$

$$g_b = g_b(w, y_b(w)). \tag{125}$$

From (3) and (124),

$$x_i(w) = y_i(w) - T_i(y_i(w)), \quad i = a, b. \tag{126}$$

Allowing for (12) and (124)–(126), the utility functions are expressed by

$$u_a(y_a(w), y_b(w)) = v_a(x_a(w)) + h_a\left(1 - \frac{y_a(w)}{w_a(w)}\right) + q_b(g_b(w, y_b(w))), \tag{127}$$

$$u_b(y_b(w)) = v_b(x_b(w)) + h_b\left(1 - \frac{y_b(w)}{w_b(w)} - g_b(w, y_b(w))\right) + q_b(g_b(w, y_b(w))). \tag{128}$$

From (127) and (128), the IC constraints are given by

$$\dot{u}_a(w) = h_a'\left(1 - \frac{y_a(w)}{w_a(w)}\right)\frac{y_a(w)w_a'(w)}{(w_a(w))^2} + q_b'(g_b(w, y_b(w)))\dot{g}_b(w, y_b(w))w_b'(w), \tag{129}$$

[15] We have used (5) to derive (121).

$$\dot{u}_b(w) = h_b'\left(1 - \frac{y_b(w)}{w_b(w)} - g_b(w, y_b(w))\right)\frac{y_b(w)w_b'(w)}{(w_b(w))^2}. \tag{130}$$

Before presenting the optimal income tax expressions, we define the elasticities of labor supply. The elasticities related to spouse b takes the same forms as (20)–(24), which can be derived from (121). The elasticities of spouse a are slightly modified. From (120),

$$\left.\frac{\partial l_a}{\partial(1 - T_a')}\right|_{u_a=\text{constant}} = -\frac{v_a'}{(1 - T_a')^2 w_a v_a'' + \frac{1}{w_a}h_a''} > 0. \tag{131}$$

Using this and (120), the compensated elasticities of labor supply are expressed by

$$\varepsilon_{lT'}^{ca}\left(\equiv \frac{1 - T_a'}{l_a}\left.\frac{\partial l_a}{\partial(1 - T_a')}\right|_{u_a=\text{constant}}\right) = -\frac{h_a'}{\left[(1 - T_a')^2 w_a^2 v_a'' + h_a''\right] l_a} > 0. \tag{132}$$

Introducing the virtual income, we obtain the following the uncompensated elasticity of labor supply:

$$\varepsilon_{lT'}^{ua}\left(\equiv \frac{1 - T_a'}{l_a}\frac{\partial l_a}{\partial(1 - T_a')}\right) = -\frac{h_a' + (1 - T_a')^2 w_a^2 l_a v_a''}{\left[(1 - T_a')^2 w_a^2 v_a'' + h_a''\right] l_a}. \tag{133}$$

From (127) and (128), the implicit function of x_i is given by

$$x_a = X_a(u_a(w), y_a(w), y_b(w)), \quad x_b = X_b(u_b(w), y_b(w)), \tag{134}$$

which satisfies

$$\frac{\partial X_a(u_a(w), y_a(w), y_b(w))}{\partial y_a(w)} = \frac{h_a'}{v_a' w_a}, \tag{135}$$

$$\frac{\partial X_b(u_b(w), y_b(w))}{\partial y_b(w)} = \frac{h_b'}{v_b' w_b}, \tag{136}$$

$$\frac{\partial X_a(u_a(w), y_a(w), y_b(w))}{\partial y_b(w)} = -\frac{q_b' g_{by_b}}{v_a'}, \tag{137}$$

$$\frac{\partial X_a(u_a(w), y_a(w), y_b(w))}{\partial u_a(w)} = \frac{1}{v_a'}, \tag{138}$$

$$\frac{\partial X_b(u_b(w), y_b(w))}{\partial u_b(w)} = \frac{1}{v_b'}. \tag{139}$$

Using (134), the government budget constraint takes the following form:

$$\int_{\underline{w}}^{\overline{w}} (y_a(w) - X_a(u_a(w), y_a(w), y_b(w)) + y_b(w) - X_b(u_b(w), y_b(w)))\pi(w)dw \geq R. \tag{140}$$

The problem for the government is to choose $y_a(w)$, $y_b(w)$, $u_a(w)$, and $u_b(w)$ to maximize its welfare function (17) subject to the budget constraint (140) and IC constraints (129) and (130). Using (106), the corresponding Lagrangian to the maximization problem is

$$
\begin{aligned}
L^n \quad \equiv \quad & \int_{\underline{w}}^{\overline{w}} \Phi(u_a(w), u_b(w))\pi(w)dw + \lambda \int_{\underline{w}}^{\overline{w}} (y_a(w) - X_a(u_a(w), y_a(w), y_b(w)) \\
& + y_b(w) - X_b(u_b(w), y_b(w)) - R)\pi(w)dw \\
& - \varsigma_a(\overline{w})u_a(\overline{w}) + \varsigma_a(\underline{w})u_a(\underline{w}) - \varsigma_b(\overline{w})u_b(\overline{w}) + \varsigma_b(\underline{w})u_b(\underline{w}) \\
& + \int_{\underline{w}}^{\overline{w}} \varsigma_a(w) \left(h_a' \left(1 - \frac{y_a(w)}{w_a(w)}\right) \frac{y_a(w)w_a'(w)}{(w_a(w))^2} \right. \\
& \left. + q_b'(g_b(w, y_b(w)))\dot{g}_b(w, y_b(w))w_b'(w) \right) dw + \int_{\underline{w}}^{\overline{w}} \dot{\varsigma}_a(w)u_a(w)dw \\
& + \int_{\underline{w}}^{\overline{w}} \varsigma_b(w)h_b' \left(1 - \frac{y_b(w)}{w_b(w)} - g_b(w, y_b(w))\right) \frac{y_b(w)w_b'(w)}{(w_b(w))^2} dw \\
& + \int_{\underline{w}}^{\overline{w}} \dot{\varsigma}_b(w)u_b(w)dw.
\end{aligned} \tag{141}
$$

Using (135)–(139), from (141) the necessary conditions are

$$
\begin{aligned}
y_a(w) : 0 = {} & \lambda\pi(w) \left(1 - \frac{h_a'\left(1 - \frac{y_a(w)}{w_a(w)}\right)}{v_a'(x_a(w))w_a(w)} \right) \\
& + \varsigma_a(w) \left(\frac{h_a'\left(1 - \frac{y_a(w)}{w_a(w)}\right)}{\left(w_a(w)\right)^2} - \frac{h_a''\left(1 - \frac{y_a(w)}{w_a(w)}\right) y_a(w)}{\left(w_a(w)\right)^2} \right) w_a'(w),
\end{aligned} \tag{142}
$$

$$y_b(w): 0 = \lambda\pi(w)\left(1 - \frac{h_b'\left(1 - \frac{y_b(w)}{w_b(w)} - g_b(w, y_b(w))\right)}{v_b'(x_b(w))w_b(w)}\right. \tag{143}$$
$$\left. + \frac{q_b'(g_b(w, y_b(w)))g_{by_b}(w, y_b(w))}{v_a'(x_a(w))}\right)$$
$$+ \varsigma_b(w)\left[\frac{h_b'\left(1 - \frac{y_b(w)}{w_b(w)} - g_b(w, y_b(w))\right)}{(w_b(w))^2}\right.$$
$$- \frac{h_b''\left(1 - \frac{y_b(w)}{w_b(w)} - g_b(w, y_b(w))\right)y_b(w)}{(w_b(w))^2}$$
$$\left. \cdot \left(\frac{1}{w_b(w)} + g_{by_b}(w, y_b(w))\right)\right] w_b'(w)$$
$$+ \varsigma_a(w)\left(q_b''(g_b(w, y_b(w)))\dot{g}_b(w, y_b(w))g_{by_b}(w, y_b(w))\right.$$
$$\left. + q_b'(g_b(w, y_b(w)))\frac{\partial \dot{g}_b(w, y_b(w))}{\partial y_b(w)}\right) w_b'(w),$$

$$u_i(w): 0 = \left(\Phi_{u_i}(u_a(w), u_b(w)) - \frac{\lambda}{v_i'(x_i(w))}\right)\pi(w) + \dot{\varsigma}_i(w), \quad i = a, b, \tag{144}$$

$$u_i(\underline{w}): 0 = \varsigma_i(\underline{w}), \quad i = a, b, \tag{145}$$

$$u_i(\overline{w}): 0 = \varsigma_i(\overline{w}), \quad i = a, b. \tag{146}$$

Applying the same procedure as that in Appendix D to (142)–(146) and utilizing (20)–(24) for $i = b$, (132), and (133), we obtain (38) and (39).

Appendix F: The Proof of Proposition 2

Individuals a and b with the same ability in different couples obtain the same income as both the production function for the household public good and utility function are separable. Let us denote $y_a(\iota) \equiv y_a(w_a(\iota))$ and $y_b(\widetilde{w}(\iota)) \equiv y_b(w_b(\widetilde{w}(\iota)))$. From (107), the first-order condition of $y_a(\iota)$ and $y_b(\widetilde{w}(\iota))$, which satisfy $y_a(\iota) = y_b(\widetilde{w}(\iota))$, is

$$\begin{aligned}
y_a(\iota),\ y_b(\widetilde{w}(\iota)) \quad : \quad 0 = {} & \lambda\pi(\iota)\left(1 - h'_a\left(1 - \frac{y_a(\iota)}{w_a(\iota)} - g_a(\iota, y_a(\iota))\right)\frac{1}{v'_a(x_a(\iota))w_a(\iota)}\right. \\
& \left.+\frac{q'_a(g_a(\iota, y_a(\iota)))g_{ay_a}(w, y_a(\iota))}{v'_b(x_v(\widetilde{w}(\iota)))}\right) \\
& + \varsigma_a(\iota)\left[h'_a\left(1 - \frac{y_a(\iota)}{w_a(\iota)} - g_a(\iota, y_a(\iota))\right)\frac{1}{(w_a(\iota))^2}\right. \\
& - h''_a\left(1 - \frac{y_a(\iota)}{w_a(\iota)} - g_a(\iota, y_a(\iota))\right)\frac{y_a(\iota)}{(w_a(\iota))^2} \\
& \left.\cdot\left(\frac{1}{w_a(\iota)} + g_{ay_a}(\iota, y_a(\iota))\right)\right] w'_a(\iota) \\
& + \varsigma_b(\widetilde{w}(\iota))\left(q''_a(g_a(\iota, y_a(\iota)))\dot{g}_a(\iota, y_a(\iota))g_{ay_a}(\iota, y_a(\iota))\right. \\
& \left.+q'_a(g_a(\iota, y_a(\iota)))\frac{\partial\dot{g}_a(\iota, y_a(\iota))}{\partial y_a(\iota)}\right) w'_a(\iota) \\
& + \lambda\pi(\widetilde{w}(\iota))\left(1 - h'_b\left(1 - \frac{y_b(\widetilde{w}(\iota))}{w_b(\widetilde{w}(\iota))} - g_b(\widetilde{w}(\iota), y_b(\widetilde{w}(\iota)))\right)\right. \\
& \cdot\frac{1}{v'_b(x_b(\widetilde{w}(\iota)))w_b(\widetilde{w}(\iota))} \\
& \left.+\frac{q'_b(g_b(\widetilde{w}(\iota), y_b(\widetilde{w}(\iota))))g_{by_b}(\widetilde{w}(\iota), y_b(\widetilde{w}(\iota)))}{v'_a(x_a(w(\iota)))}\right) \\
& + \varsigma_b(\widetilde{w}(\iota))\left[h'_b\left(1 - \frac{y_b(\widetilde{w}(\iota))}{w_b(\widetilde{w}(\iota))} - g_b(\widetilde{w}(\iota), y_b(\widetilde{w}(\iota)))\right)\frac{1}{(w_b(\widetilde{w}(\iota)))^2}\right. \\
& - h''_b\left(1 - \frac{y_b(\widetilde{w}(\iota))}{w_b(\widetilde{w}(\iota))} - g_b(\widetilde{w}(\iota), y_b(\widetilde{w}(\iota)))\right)\frac{y(\widetilde{w}(\iota))}{(w_b(\widetilde{w}(\iota)))^2} \\
& \left.\cdot\left(\frac{1}{w_b(\widetilde{w}(\iota))} + g_{by_b}(\widetilde{w}(\iota), y_b(\widetilde{w}(\iota)))\right)\right] w'_b(\widetilde{w}(\iota)) \\
& + \varsigma_a(\iota)\left(q''_b(g_b(\widetilde{w}(\iota), y_b(\widetilde{w}(\iota))))\right. \\
& \cdot \dot{g}_b(\widetilde{w}(\iota), y_b(\widetilde{w}(\iota)))g_{by_b}(\widetilde{w}(\iota), y_b(\widetilde{w}(\iota))) \\
& \left.+q'_b(g_b(\widetilde{w}(\iota), y_b(\widetilde{w}(\iota))))\frac{\partial\dot{g}_b(\widetilde{w}(\iota), y_b(\widetilde{w}(\iota)))}{\partial y_b(\widetilde{w}(\iota))}\right) w'_b(\widetilde{w}(\iota)).
\end{aligned} \tag{147}$$

Note that as the conditions (109)–(111) hold even in the case without tagging, (112)–(114) also hold. Furthermore, note that, from (9), the following holds:

$$1 - T'(y_i) = \frac{h'_a}{v'_a w_a} = \frac{h'_b}{v'_b w_b}, \quad i = a, b, \tag{148}$$

for $y_a = y_b$.

Applying (5), (7), (8), (9), (114), and (148) to (147) yields

$$\begin{aligned}
\frac{T'(\iota)}{1-T'(\iota)} = {} & \frac{\pi(\iota)}{\pi(\iota)+\pi(\widetilde{w}(\iota))}\left\{\int_{\iota}^{\overline{w}}\left(\frac{1}{v_a'(x_a(s))}-\frac{\Phi_{u_a}(u_a(s),u_b(s))}{\lambda}\right)\pi(s)ds\right. \\
& \cdot\frac{v_a'(x_a(\iota))w_a'(\iota)}{\pi(\iota)w_a(\iota)}\left[1-\frac{h_a''(\cdot)y_a(\iota)}{h_a'(\cdot)}\left(\frac{1}{w_a(\iota)}+g_{ay_a}(\cdot)\right)\right. \\
& \left.-\frac{w_a(\iota)g_{ay_a}(\cdot)v_a'(x_a(\iota))}{v_b'(x_b(\widetilde{w}(\iota)))}\right] \\
& -\int_{\widetilde{w}(\iota)}^{\overline{w}}\left(\frac{1}{v_b'(x_b(s))}-\frac{\Phi_{u_b}(u_a(s),u_b(s))}{\lambda}\right)\pi(s)ds \\
& \left.\cdot\frac{v_a'(x_a(\iota))g_{ay_a}(\cdot)w_a'(\iota)}{\pi(\iota)}\left(\frac{y_a(\iota)}{q_a'(\cdot)}\frac{\partial q_a'(\cdot)}{\partial y_a(\iota)}+\frac{y_a(\iota)}{\dot{g}_a(\cdot)}\frac{\partial\dot{g}_a(\cdot)}{\partial y_a(\iota)}\right)\right\} \\
& +\frac{\pi(\widetilde{w}(\iota))}{\pi(w)+\pi(\widetilde{w}(\iota))}\left\{\int_{\widetilde{w}(\iota)}^{\overline{w}}\left(\frac{1}{v_b'(x_b(s))}-\frac{\Phi_{u_b}(u_a(s),u_b(s))}{\lambda}\right)\pi(s)ds\right. \\
& \cdot\frac{v_b'(x_b(\widetilde{w}(\iota)))w_b'(\widetilde{w}(\iota))}{\pi(\widetilde{w}(\iota))w_b(\widetilde{w}(\iota))} \\
& \cdot\left[1-\frac{h_b''(\cdot)y_b(\widetilde{w}(\iota))}{h_b'(\cdot)}\left(\frac{1}{w_b(\widetilde{w}(\iota))}+g_{by_b}(\cdot)\right)\right. \\
& \left.-\frac{w_b(\widetilde{w}(\iota))g_{by_b}(\cdot)v_b'(x_b(\widetilde{w}(\iota)))}{v_a'(x_a(\iota))}\right] \\
& -\int_{\iota}^{\overline{w}}\left(\frac{1}{v_a'(x_a(s))}-\frac{\Phi_{u_a}(u_a(s),u_b(s))}{\lambda}\right)\pi(s)ds \\
& \cdot\frac{v_b'(x_b(\widetilde{w}(\iota)))g_{by_b}(\cdot)w_b'(\widetilde{w}(\iota))}{\pi(\widetilde{w}(\iota))} \\
& \left.\cdot\left(\frac{y_b(\widetilde{w}(\iota))}{q_b'(\cdot)}\frac{\partial q_b'(\cdot)}{\partial y_b(\widetilde{w}(\iota))}+\frac{y_b(\widetilde{w}(\iota))}{\dot{g}_b(\cdot)}\frac{\partial\dot{g}_b(\cdot)}{\partial y_b(\widetilde{w}(\iota))}\right)\right\}.
\end{aligned} \tag{149}$$

Applying (117) and (118) to (149) and making use of the definitions of A_i^n, B_i^n, C_i^n, D_i^n, and E_i^n yields (40).

Appendix G: The Proof of Proposition 3

The government must allow for the IC constraints of both the cooperative and non-cooperative couples, that is, (19) and (86). Thus, the Lagrangian for the optimization problem is

$$
\begin{aligned}
L^{cn} \equiv & \int_{\underline{w}}^{\overline{w}} \left[(u_a^n(w) + u_b^n(w))\gamma(w) + (u_a^c(w) + u_b^c(w))(1-\gamma(w)) \right] \pi(w)dw \\
& + \lambda \int_{\underline{w}}^{\overline{w}} [(y_a^n(w) + y_b^n(w) - X_a^n(u_a^n(w), y_a^n(w), y_b^n(w)) \\
& - X_b^n(u_b^n(w), y_b^n(w), y_a^n(w)))\gamma(w) \\
& + (y_a^c(w) + y_b^c(w) - X_a^c(u_a^c(w), y_a^c(w), y_b^c(w)) \\
& - X_b^c(u_b^c(w), y_b^c(w), y_a^c(w)))(1-\gamma(w)) - R]\pi(w)dw \\
& - \varsigma_a^n(\overline{w})u_a^n(\overline{w}) + \varsigma_a^n(\underline{w})u_a^n(\underline{w}) - \varsigma_b^n(\overline{w})u_b^n(\overline{w}) + \varsigma_b^n(\underline{w})u_b^n(\underline{w}) \\
& - \varsigma^c(\overline{w})u^c(\overline{w}) + \varsigma^c(\underline{w})u^c(\underline{w}) \\
& + \int_{\underline{w}}^{\overline{w}} \varsigma_a^n(w) \left[h_a^{n\prime} \left(1 - \frac{y_a^n(w)}{w_a(w)} - g_a^n(w, y_a^n(w)) \right) \frac{y_a^n(w)w_a'(w)}{(w_a(w))^2} \right. \\
& \left. + q_b^{n\prime}(g_b^n(w, y_b^n(w)))\dot{g}_b^n(w, y_b^n(w))w_b'(w) \right] dw + \int_{\underline{w}}^{\overline{w}} \dot{\varsigma}_a^n(w)u_a^n(w)dw \\
& + \int_{\underline{w}}^{\overline{w}} \varsigma_b^n(w) \left[h_b^{n\prime} \left(1 - \frac{y_b^n(w)}{w_b(w)} - g_b^n(w, y_b^n(w)) \right) \frac{y_b^n(w)w_b'(w)}{(w_b(w))^2} \right. \\
& \left. + q_a^{n\prime}(g_a^n(w, y_a^n(w)))\dot{g}_a^n(w, y_a^n(w))w_a'(w) \right] dw + \int_{\underline{w}}^{\overline{w}} \dot{\varsigma}_b^n(w)u_b^n(w)dw \\
& + \int_{\underline{w}}^{\overline{w}} \varsigma^c(w) \left[h_a^{c\prime} \left(1 - \frac{y_a^c(w)}{w_a(w)} - g_a^c(w, y_a^c(w)) \right) \frac{y_a^c(w)w_a'(w)}{(w_a(w))^2} \right. \\
& \left. + h_b^{c\prime} \left(1 - \frac{y_b^c(w)}{w_b(w)} - g_b^c(w, y_b^c(w)) \right) \frac{y_b^c(w)w_b'(w)}{(w_b(w))^2} \right] dw + \int_{\underline{w}}^{\overline{w}} \dot{\varsigma}^c(w)u^c(w)dw.
\end{aligned}
$$

Then, the FOC with respect to y_i^c and y_i^n that satisfy that $y_i^n(w) = y_i^c(\widehat{w}(w))$ are

$$
\begin{aligned}
y_i^n(w), y_i^c(\widehat{w}(w)) : 0 = \lambda\gamma(w)\pi(w) \Bigg(1 - & \frac{h_i^{n\prime}\left(1 - \frac{y_i^n(w)}{w_i(w)} - g_i^n(w, y_i^n(w))\right)}{v_i^{n\prime}(x_i^n(w)) w_i(w)} \\
& + \frac{q_i^{n\prime}(g_i^n(w, y_i^n(w))) g_{iy_i}^n(w, y_i^n(w))}{v_j^{n\prime}(x_j^n(w))} \Bigg) \\
& + \varsigma_i^n(w) \Bigg[\frac{h_i^{n\prime}\left(1 - \frac{y_i^n(w)}{w_i(w)} - g_i^n(w, y_i^n(w))\right)}{(w_i(w))^2} \\
& - \frac{h_i^{n\prime\prime}\left(1 - \frac{y_i^n(w)}{w_i(w)} - g_i^n(w, y_i^n(w))\right) y_i^n(w)}{(w_i(w))^2} \\
& \cdot \left(\frac{1}{w_i(w)} + g_{iy_i}^n(w, y_i^n(w)) \right) \Bigg] w_i'(w) \\
& + \varsigma_j^n(w) \Bigg(q_i^{n\prime\prime}(g_i^n(w, y_i^n(w))) \dot{g}_i^n(w, y_i^n(w)) g_{iy_i}^n(w, y_i^n(w)) \\
& + q_i^{n\prime}(g_i^n(w, y_i^n(w))) \frac{\partial \dot{g}_i^n(w, y_i^n(w))}{\partial y_i^n(w)} \Bigg) w_i'(w) \\
& + \lambda \cdot (1 - \gamma(\widehat{w}(w))) \pi(\widehat{w}(w)) \\
& \cdot \left(1 - \frac{h_i^{c\prime}\left(1 - \frac{y_i^c(\widehat{w}(w))}{w_i(\widehat{w}(w))} - g_i^c(\widehat{w}(w), y_i^c(\widehat{w}(w)))\right)}{v_i^{c\prime}(x_i^c(\widehat{w}(w))) w_i(\widehat{w}(w))} \right) \\
& + \varsigma^c(\widehat{w}(w)) \Bigg[\frac{h_i^{c\prime}\left(1 - \frac{y_i^c(\widehat{w}(w))}{w_i(\widehat{w}(w))} - g_i^c(\widehat{w}(w), y_i^c(\widehat{w}(w)))\right)}{(w_i(\widehat{w}(w)))^2} \\
& - \frac{h_i^{c\prime\prime}\left(1 - \frac{y_i^c(\widehat{w}(w))}{w_i(\widehat{w}(w))} - g_i^c(\widehat{w}(w), y_i^c(\widehat{w}(w)))\right) y_i^c(\widehat{w}(w))}{(w_i(\widehat{w}(w)))^2} \\
& \cdot \left(\frac{1}{w_i(\widehat{w}(w))} + g_{iy_i}^c(\widehat{w}(w), y_i^c(\widehat{w}(w))) \right) \Bigg] w_i'(\widehat{w}(w)), \\
& \qquad\qquad\qquad i, j = a, b, \ i \neq j,
\end{aligned} \tag{150}
$$

$$
u_i^n(w) : 0 = \gamma(w)\pi(w) \left(1 - \frac{\lambda}{v_i^{n\prime}(x_i^n(w))} \right) + \dot{\varsigma}_i^n(w), \quad i = a, b, \tag{151}
$$

$$
u_i^c(\widehat{w}(w)) : 0 = (1 - \gamma(\widehat{w}(w))) \pi(\widehat{w}(w)) \left(1 - \frac{\lambda}{v_i^{c\prime}(x_i^c(\widehat{w}(w)))} \right) + \dot{\varsigma}_i^c(\widehat{w}(w)), \quad i = a, b, \tag{152}
$$

$$
u_i^n(\overline{w}) : 0 = \varsigma_i^n(\overline{w}), \quad i = a, b, \tag{153}
$$

$$u_i^c(\widehat{w}(\overline{w})) : 0 = \varsigma_i^c(\widehat{w}(\overline{w})), \quad i = a, b, \tag{154}$$

$$u_i^n(\underline{w}) : 0 = \varsigma_i^n(\underline{w}), \quad i = a, b, \tag{155}$$

$$u_i^c(\widehat{w}(\underline{w})) : 0 = \varsigma_i^c(\widehat{w}(\underline{w})), \quad i = a, b. \tag{156}$$

From (9), we have

$$1 - T_i'(y_i^j) = \frac{h_i^{n\prime}}{v_i^{n\prime} w_i(w)} = \frac{h_i^{c\prime}}{v_i^{c\prime} w_i(\widehat{w}(w))}, \quad i = a, b, \quad j = c, n, \tag{157}$$

for $y_i^n(w) = y_i^c(\widehat{w}(w))$. Using (157), (150) can be rewritten as

$$\begin{aligned}
&[\gamma(w)\pi(w) + (1 - \gamma(\widehat{w}(w)))\pi(\widehat{w}(w))] \frac{T_i'(w)}{1 - T_i'(w)} \\
&= -\frac{\varsigma_i^n(w)}{\lambda} \frac{v_i^{n\prime}(x_i^n(w))}{w_i'(w)} \\
&\quad \cdot \left[1 - \frac{h_i^{n\prime\prime}\left(1 - \frac{y_i^n(w)}{w_i(w)} - g_i^n(w, y_i^n(w))\right) y_i^n(w)}{h_i^{n\prime}\left(1 - \frac{y_i^n(w)}{w_i(w)} - g_i^n(w, y_i^n(w))\right)} \left(\frac{1}{w_i'(w)} + g_{iy_i}^n(w, y_i^n(w))\right)\right] \\
&\quad - \frac{\varsigma_j^n(w)}{\lambda} \frac{v_i^{n\prime}(x_i^n(w)) w_i'(w)}{h_i^{n\prime}\left(1 - \frac{y_i^n(w)}{w_i(w)} - g_i^n(w, y_i^n(w))\right)} \\
&\quad \cdot \left(q_i^{n\prime\prime}(g_i^n(w, y_i^n(w))) \dot{g}_i^n(w, y_i^n(w)) g_{iy_i}^n(w, y_i^n(w))\right. \\
&\quad \left. + q_i^{n\prime}(g_i^n(w, y_i^n(w))) \frac{\partial \dot{g}_i^n(w, y_i^n(w))}{\partial y_i^n(w)}\right) \\
&\quad - \gamma(w)\pi(w) \frac{v_i^{n\prime}(x_i^n(w)) w_i(w)}{h_i^{n\prime}\left(1 - \frac{y_i^n(w)}{w_i(w)} - g_i^n(w, y_i^n(w))\right)} \frac{q_i^{n\prime}(g_i^n(w, y_i^n(w))) g_{iy_i}^n(w, y_i^n(w))}{v_j^{n\prime}(x_j^n(w))} \\
&\quad - \frac{\varsigma^c(\widehat{w}(w))}{\lambda} \frac{v_i^{c\prime}(x_i^c(\widehat{w}(w))) w_i'(\widehat{w}(w))}{w_i(\widehat{w}(w))} \\
&\quad \cdot \left[1 - \frac{h_i^{c\prime\prime}\left(1 - \frac{y_i^c(\widehat{w}(w))}{w_i(\widehat{w}(w))} - g_i^c(\widehat{w}(w), y_i^c(\widehat{w}(w)))\right) y_i^c(\widehat{w}(w))}{h_i^{c\prime}\left(1 - \frac{y_i^c(\widehat{w}(w))}{w_i(\widehat{w}(w))} - g_i^c(\widehat{w}(w), y_i^c(\widehat{w}(w)))\right)}\right. \\
&\quad \left. \cdot \left(\frac{1}{w_i(\widehat{w}(w))} + g_{iy_i}^c(\widehat{w}(w), y_i^c(\widehat{w}(w)))\right)\right], \quad i, j = a, b, \ i \neq j.
\end{aligned} \tag{158}$$

From (151)–(155), we obtain

$$-\frac{\varsigma_i^n(w)}{\lambda} = \int_w^{\overline{w}} \left(\frac{1}{v_i^{n\prime}(x_i^n(s))} - \frac{1}{\lambda}\right) \gamma(s)\pi(s) ds, \quad i = a, b, \tag{159}$$

$$-\frac{\varsigma_i^c(\widehat{w}(w))}{\lambda} = \int_{\widehat{w}(w)}^{\overline{w}} \left(\frac{1}{v_i^{c\prime}(x_i^c(s))} - \frac{1}{\lambda} \right) (1-\gamma(s))\pi(s)ds, \quad i = a, b. \tag{160}$$

Using (5), (7), (8), (160), (159), and the definitions of the elasticities and θ, (158) leads to (45).

Appendix H: The Proof of Proposition 4

From (47) and (48), the FOCs of the utility maximization with respect to g_i and z_i are

$$g_i : 0 = -h_i' \left(1 - \frac{y_i}{w_i} - g_i \right) + q_i'(g_i + r_i(z_i)), \quad i = a, b, \tag{161}$$

$$z_i : 0 = -(1+t) + q_i'(g_i + r_i(z_i))r_i'(z_i), \quad i = a, b. \tag{162}$$

These two equations yield

$$g_i(w_i, t, y_i), \quad z_i(w_i, t, y_i), \quad i = a, b, \tag{163}$$

which satisfy that

$$g_{iy_i} \left(\equiv \frac{\partial g_i}{\partial y_i} \right) = -\frac{[q_i'' \cdot (r_i')^2 + q_i' r_i''] \, h_i''}{\{[q_i'' \cdot (r_i')^2 + q_i' r_i''] \, h_i'' + q_i'' q_i' r_i''\} \, w_i} < 0, \quad i = a, b, \tag{164}$$

$$z_{iy_i} \left(\equiv \frac{\partial z_i}{\partial y_i} \right) = \frac{h_i'' q_i'' r_i'}{\{[q_i'' \cdot (r_i')^2 + q_i' r_i''] \, h_i'' + q_i'' q_i' r_i''\} \, w_i} > 0, \quad i = a, b, \tag{165}$$

$$g_{it} \left(\equiv \frac{\partial g_i}{\partial t} \right) = -\frac{q_i'' r_i'}{\{[q_i'' \cdot (r_i')^2 + q_i' r_i''] \, h_i'' + q_i'' q_i' r_i''\} \, w_i} > 0, \quad i = a, b, \tag{166}$$

$$z_{it} \left(\equiv \frac{\partial z_i}{\partial t} \right) = \frac{h_i'' + q_i''}{\{[q_i'' \cdot (r_i')^2 + q_i' r_i''] \, h_i'' + q_i'' q_i' r_i''\} \, w_i} < 0, \quad i = a, b, \tag{167}$$

$$\dot{g}_i \left(\equiv \frac{\partial g_i}{\partial w_i} \right) = \frac{[q_i'' \cdot (r_i')^2 + q_i' r_i''] \, h_i'' y_i}{\{[q_i'' \cdot (r_i')^2 + q_i' r_i''] \, h_i'' + q_i'' q_i' r_i''\} \, w_i^2} > 0, \quad i = a, b, \tag{168}$$

$$\dot{z}_i \left(\equiv \frac{\partial z_i}{\partial w_i} \right) = \frac{h_i'' q_i'' r_i' y_i}{\{[q_i'' \cdot (r_i')^2 + q_i' r_i''] \, h_i'' + q_i'' q_i' r_i''\} \, w_i^2} > 0, \quad i = a, b. \tag{169}$$

From (166) and (167), we have

$$q_{it} = q_i' \cdot (g_{it} + r_i' z_{it}) = \frac{q_i' r_i' h_i''}{\{[q_i'' \cdot (r_i')^2 + q_i' r_i''] \, h_i'' + q_i'' q_i' r_i''\} \, w_i} < 0, \quad i = a, b. \tag{170}$$

Using (163), the utility function is rewritten as

$$\begin{aligned} u_i = {} & y_i - T_i(y_i) - (1+t) z_i(w_i, t, y_i) + h_i \left(1 - \frac{y_i}{w_i} - g_i(w_i, t, y_i) \right) \\ & + q_a(g_a(w_a, t, y_a) + r_a(z_a(w_a, t, y_a))) \\ & + q_b \left(g_b(w_b, t, y_b) + r_b(z_b(w_b, t, y_b)) \right), \quad i = a, b. \end{aligned} \tag{171}$$

Allowing for (161) and (162), the FOC with respect to y_i for utility maximization is

$$y_i : 0 = 1 - T_i'(y_i) - h_i'\left(1 - \frac{y_i}{w_i} - g_i(w_i, t, y_i)\right)\frac{1}{w_i}, \quad i = a, b, \tag{172}$$

which yields[16]

$$y_i(w_i), \quad i = a, b. \tag{173}$$

Allowing for (12), from (163) and (173) we have (49).

Before presenting the optimal marginal tax rate, let us define the elasticities as follows. Using (172) again, we obtain the elasticity of labor supply:

$$\varepsilon^i_{lT'}\left(\equiv \frac{1 - T_i'}{l_i}\frac{\partial l_i}{\partial(1 - T_i')}\right) = -\frac{h_i'}{(1 + w_i g_{iy_i})\, h_i'' l_i} > 0, \quad i = a, b. \tag{174}$$

Note that it follows that $dg_i/d(1 - T_i') = w_i g_{iy_i} \cdot (dl_i/d(1 - T_i'))$ and $dz_i/d(1 - T_i') = w_i z_{iy_i} \cdot (dl_i/d(1 - T_i'))$.[17] Using these and (198), we can define the elasticities of housework time and commodity, respectively, as

$$\varepsilon^i_{gT'}\left(\equiv -\frac{1 - T_i'}{g_i}\frac{\partial g_i}{\partial(1 - T_i')}\right) = \frac{w_i g_{iy_i} h_i'}{(1 + w_i g_{iy_i}) h_i'' g_i} > 0, \quad i = a, b, \tag{175}$$

$$\varepsilon^i_{zT'}\left(\equiv \frac{1 - T_i'}{z_i}\frac{\partial z_i}{\partial(1 - T_i')}\right) = -\frac{w_i z_{iy_i} h_i'}{(1 + w_i g_{iy_i}) h_i'' z_i} > 0, \quad i = a, b. \tag{176}$$

The following elasticities are straightforwardly obtained from the functions $q_i'(\cdot)$, $r_i'(\cdot)$, $\dot{g}_i(\cdot)$, and $\dot{z}_i(\cdot)$:

$$\varepsilon^i_{q'y}\left(\equiv \frac{y_i}{q_i'}\frac{\partial q_i'}{\partial y_i}\right) = \frac{y_i q_i'' \cdot (g_{iy_i} + r_i' z_{iy_i})}{q_i'} > 0, \quad i = a, b, \tag{177}$$

$$\varepsilon^i_{r'y}\left(\equiv -\frac{y_i}{r_i'}\frac{\partial r_i'}{\partial y_i}\right) = -\frac{y_i r_i'' z_{iy_i}}{r_i'} > 0, \quad i = a, b, \tag{178}$$

$$\begin{aligned}
\varepsilon^i_{\dot{g}y}\left(\equiv \frac{y_i}{\dot{g}_i}\frac{\partial \dot{g}_i}{\partial y_i}\right) \quad = \quad & \left\{\frac{1}{[(q_i'' r_i'^2 + q_i' r_i'') h_i'' + q_i'' q_i' r_i''] \, w_i^2}\right\} \\
& \cdot \{(h_i'')^2 (q_i'' r_i'^2 + q_i' r_i'')^2 + h_i' q_i'' q_i' r_i'' \cdot (q_i'' r_i'^2 + q_i' r_i'') \\
& - q_i^3 h_i'' y_i \cdot (2 r_i''^2 - r_i'^2 r_i''' - r_i'^2 r_i'')(g_{iy_i} + r_i' z_{iy_i}) \\
& + (g_{iy_i} + r_i' z_{iy_i}) q_i''^2 h_i'' y_i r_i''^2 q_i'^2 \\
& \cdot \left[\frac{(q_i'' r_i'^2 + q_i' r_i'') h_i'''}{r_i'' q_i' h_i''^2} - \frac{q_i'''}{q_i''^2}\right]\Bigg\}, \quad i = a, b,
\end{aligned} \tag{179}$$

[16] Strictly speaking, it should be expressed as $y_i(w_i, t)$. However, as y_i is a control variable in our analysis, t is not denoted as an argument of the function y_i to simplify the symbol method.

[17] Note that $w_i = dy_i/dl_i$.

$$
\begin{aligned}
\varepsilon^i_{\dot{z}y}\left(\equiv \frac{y_i}{\dot{z}_i}\frac{\partial \dot{z}_i}{\partial y_i}\right) \quad = \quad & \left\{\frac{1}{\left[(q_i'' r_i'^2 + q_i' r_i'')h_i'' + q_i'' q_i' r_i''\right] w_i^2}\right\} \\
& \cdot \{(h_i'')^2 q_i'' r_i' \cdot (q_i'' r_i'^2 + q_i' r_i'') + h_i'' q_i'' r_i' r_i'' q_i' \\
& + h_i''^2 q_i'' r_i'' y_i z_{iy_i} \cdot (q_i'' r_i'^2 + q_i' r_i'') + h_i'' q_i''^2 q_i' r_i''^2 y_i z_{iy_i} \\
& + h_i''^2 r_i' y_i q_i''' \cdot (g_{iy_i} + r_i' z_{iy_i}) + q_i''^2 q_i' r_i'' r_i' h_i''' y_i \cdot \left(-\frac{1}{w_i} - g_{iy_i}\right) \\
& + -h_i''^2 q_i'' r_i' y_i \cdot [q_i''' r_i'^2 (g_{iy_i} + r_i' z_{iy_i}) + q_i'' 2 r_i' r_i'' z_{iy_i} \\
& + q_i'' r_i'' \cdot (g_{iy_i} + r_i' z_{iy_i}) + q_i' r_i''' z_{iy_i}] \\
& - h_i'' q_i''^3 r_i'' r_i' y_i \cdot (g_{iy_i} + r_i' z_{iy_i}) - h_i'' q_i''^2 r_i' y_i r_i''' z_{iy_i} \} , \quad i = a, b.
\end{aligned} \tag{180}
$$

Substituting (49) into (171) and utilizing (161), (162), and (172), we obtain

$$
\begin{aligned}
\dot{u}_i(w) = & \, h_i' \left(1 - \frac{y_i(w)}{w_i(w)} - g_i(w, t, y_i(w))\right) \frac{y_i(w) w_i'(w)}{(w_i(w))^2} \\
& + q_j'(w) \left(\dot{g}_j(w, t, y_i(w)) + r_j'(z_i(w, t, y_i(w))) \dot{z}_j(w, t, y_i(w))\right) w_j'(w), \\
& \qquad i, j = a, b, \ i \neq j.
\end{aligned} \tag{181}
$$

Note that the following holds:

$$
\int_{\underline{w}}^{\overline{w}} \varsigma_i(w) \dot{u}_i(w) dw = \varsigma_i(\overline{w}) u_i(\overline{w}) - \varsigma_i(\underline{w}) u_i(\underline{w}) - \int_{\underline{w}}^{\overline{w}} \dot{\varsigma}_i(w) u_i(w) dw, \quad i = a, b. \tag{182}
$$

From (47) and (49), we have the implicit function of x_i with respect to $u_i(w)$, $y_a(w)$, $y_b(w)$, and t :

$$
x_i = X_i(u_i(w), y_i(w), y_j(w), t), \quad i, j = a, b, \ i \neq j, \tag{183}
$$

which satisfies

$$
\frac{\partial X_i(u_i(w), y_i(w), y_j(w), t)}{\partial y_i(w)} = \frac{h_i'}{w_i} - q_i' r_i' z_{iy_i}, \quad i, j = a, b, \ i \neq j, \tag{184}
$$

$$
\frac{\partial X_i(u_i(w), y_i(w), y_j(w), t)}{\partial y_j(w)} = -q_j' \cdot (g_{jy_j} + r_j' z_{jy_j}), \quad i, j = a, b, \ i \neq j, \tag{185}
$$

$$
\frac{\partial X_i(u_i(w), y_i(w), y_j(w), t)}{\partial u_i(w)} = 1, \quad i, j = a, b, \ i \neq j, \tag{186}
$$

$$
\frac{\partial X_i(u_i(w), y_i(w), y_j(w), t)}{\partial t} = -q_i' r_i' z_{it} - q_j' \cdot \left(g_{jt} + r_j' z_{jt}\right), \quad i, j = a, b, \ i \neq j, \tag{187}
$$

where (161) is used to derive (184)–(187).

Using (182) and (183), the Lagrangian for the optimization problem is

$$
\begin{aligned}
L^z \equiv & \int_{\underline{w}}^{\overline{w}} \Phi(u_a(w), u_b(w))\pi(w)dw \qquad (188)\\
& + \lambda \int_{\underline{w}}^{\overline{w}} (y_a(w) - X_a(u_a(w), y_a(w), y_b(w), t) - z_a(w, t, y_a(w)) \\
& + y_b(w) - X_b(u_b(w), y_b(w), y_a(w), t) - z_b(w, t, y_b(w)) - R)\pi(w)dw \\
& - \varsigma_a(\overline{w})u_a(\overline{w}) + \varsigma_a(\underline{w})u_a(\underline{w}) - \varsigma_b(\overline{w})u_b(\overline{w}) + \varsigma_b(\underline{w})u_b(\underline{w}) \\
& + \int_{\underline{w}}^{\overline{w}} \varsigma_a(w) \left[h_a' \left(1 - \frac{y_a(w)}{w_a(w)} - g_a(w, t, y_a(w)) \right) \frac{y_a(w)w_a'(w)}{(w_a(w))^2} \right. \\
& + q_b'(g_b(w, t, y_b(w))) \left(\dot{g}_b(w, t, y_b(w)) \right. \\
& \left. \left. + r_b'(z_b(w, t, y_b(w)))\dot{z}_b(w, t, y_b(w)) \right) w_b'(w) \right] dw + \int_{\underline{w}}^{\overline{w}} \dot{\varsigma}_a(w)u_a(w)dw \\
& + \int_{\underline{w}}^{\overline{w}} \varsigma_b(w) \left[h_b' \left(1 - \frac{y_b(w)}{w_b(w)} - g_b(w, t, y_b(w)) \right) \frac{y_b(w)w_b'(w)}{(w_b(w))^2} \right. \\
& + q_a'(g_b(w, t, y_b(w))) \left(\dot{g}_a(w, t, y_b(w)) \right. \\
& \left. + r_a'(z_a(w, t, y_a(w))\dot{z}_a(w, t, y_a(w))w_a'(w) \right] dw + \int_{\underline{w}}^{\overline{w}} \dot{\varsigma}_b(w)u_b(w)dw.
\end{aligned}
$$

We will omit t as an argument from now on. The necessary conditions (assuming an interior solution) are

$$
\begin{aligned}
y_i(w) : 0 = \lambda\pi(w) \Bigg[1 - \frac{h_i'\left(1 - \frac{y_i(w)}{w_i(w)} - g_i(w, y_i(w))\right)}{w_i(w)} \\
&+q_i'(w) r_i'(z_i(w, y_i(w))) z_{iy_i}(w, y_i(w)) \\
&+q_i'(w) \left(g_{iy_i}(w, y_i(w)) + r_i'(z_i(w, y_i(w))) z_{iy_i}(w, y_i(w))\right) \\
&-z_{iy_i}(w, y_i(w))\Bigg] \\
&+\varsigma_i(w) \Bigg[\frac{h_i'\left(1 - \frac{y_i(w)}{w_i(w)} - g_i(w, y_i(w))\right)}{(w_i(w))^2} \\
&-\frac{h_i''\left(1 - \frac{y_i(w)}{w_i(w)} - g_i(w, y_i(w))\right) y_i(w)}{(w_i(w))^2} \\
&\cdot \left(\frac{1}{w_i(w)} + g_{iy_i}(w, y_i(w))\right)\Bigg] w_i'(w) \\
&+ \varsigma_j(w) q_i''(w) \left(\dot{g}_i(w, y_i(w)) + r_i'(z_i(w, y_i(w))) \dot{z}_i(w, y_i(w))\right) \\
&\cdot (g_{iy_i}(w, y_i(w)) + r_i'(z_i(w, y_i(w))) z_{iy_i}(w, y_i(w))) w_i'(w) \\
&+ \varsigma_j(w) q_i'(w) \left(\frac{\partial \dot{g}_i(w, y_i(w))}{\partial y_i(w)} + r_i'(z_i(w, y_i(w))) \frac{\partial \dot{z}_i(w, y_i(w))}{\partial y_i(w)} \right. \\
&\left. + r_i''(z_i(w, y_i(w))) \dot{z}_i(w, y_i(w)) z_{iy_i}(w, y_i(w)) \right) w_i'(w), \\
&\qquad i, j = a, b,\ i \neq j.
\end{aligned}
\tag{189}
$$

$$
\begin{aligned}
t \;\;:\;\; 0 = \lambda \int_{\underline{w}}^{\overline{w}} & [q_a'(w) r_a'(z_a(w, y_a(w))) z_{at}(w, y_a(w)) \\
& + q_b'(w) \left(g_{bt}(w, y_b(w)) + r_b'(z_b(w, y_b(w))) z_{bt}(w, y_b(w)) \right) - z_{at}(w, y_a(w))) \\
& + q_b'(w) r_b'(z_b(w, y_b(w))) z_{bt}(w, y_b(w)) \\
& + q_a'(w) \left(g_{at}(w, y_a(w)) + r_a'(z_a(w, y_b(w))) z_{at}(w, y_a(w)) \right) \\
& - z_{bt}(w, y_b(w)))] \pi(w) dw + \int_{\underline{w}}^{\overline{w}} \varsigma_a(w) \left\{ -h_a'' \left(1 - \frac{y_a(w)}{w_a(w)} - g_a(w, y_a(w)) \right) \right. \\
& \cdot g_{at}(w, y_a(w)) \frac{y_a(w) w_a'(w)}{(w_a(w))^2} \\
& + \frac{\partial \left[q_b'(w) \left(\dot{g}_b(w, y_b(w)) + r_b'(z_b(w, y_b(w))) \dot{z}_b(w, y_b(w)) \right) w_b'(w) \right]}{\partial t} \\
& + \int_{\underline{w}}^{\overline{w}} \varsigma_b(w) \left\{ -h_b'' \left(1 - \frac{y_b(w)}{w_b(w)} - g_b(w, y_b(w)) \right) g_{bt}(w, y_b(w)) \frac{y_b(w) w_b'(w)}{(w_b(w))^2} \right. \\
& \left. + \frac{\partial \left[q_a'(w) \left(\dot{g}_a(w, y_a(w)) + r_a'(z_a(w, y_a(w))) \dot{z}_a(w, y_b(w)) \right) w_a'(w) \right]}{\partial t} \right\} dw,
\end{aligned} \tag{190}
$$

$$
u_i(w) : 0 = \left(\Phi_{u_i}(u_a(w), u_b(w)) - \lambda \right) \pi(w) + \dot{\varsigma}_i(w), \quad i = a, b, \tag{191}
$$

$$
u_i(\underline{w}) : 0 = \varsigma_i(\underline{w}), \quad i = a, b, \tag{192}
$$

$$
u_i(\overline{w}) : 0 = \varsigma_i(\overline{w}), \quad i = a, b. \tag{193}
$$

Integrating $\dot{\varsigma}_i(w)$ and making use of (193) yields

$$
\int_w^{\overline{w}} \dot{\varsigma}_i(w) dw = \varsigma_i(\overline{w}) - \varsigma_i(w) = -\varsigma_i(w), \quad i = a, b. \tag{194}
$$

Integrating (191) yields

$$
0 = \int_w^{\overline{w}} \left(\Phi_{u_i}(u_a(s), u_b(s)) - \lambda \right) \pi(s) ds + \int_w^{\overline{w}} \dot{\varsigma}_i(s) ds, \quad i = a, b. \tag{195}
$$

(194) and (195) yield

$$
\frac{\varsigma_i(w)}{\lambda} = - \int_w^{\overline{w}} \left(1 - \frac{\Phi_{u_i}(u_a(s), u_b(s))}{\lambda} \right) \pi(s) ds, \quad i = a, b. \tag{196}
$$

Using (161), (162), (164), (165), (168), (169), (172), and (196), after some manipulations, (189) leads to,

$$\begin{aligned}
\frac{T_i'(w)}{1-T_i'(w)} = & \int_w^{\overline{w}} \left(1-\frac{\Phi_{u_i}(u_a(s),u_b(s))}{\lambda}\right)\pi(s)ds\frac{1}{\pi(w)w_i(w)} \\
& \cdot\left[1-\frac{h_i''(\cdot)y_i(w)}{h_i'(\cdot)}\left(\frac{1}{w_i(w)}+g_{iy_i}(\cdot)\right)\right]w_i'(w) \\
& -\frac{t}{1+t}w_i(w)r_i'(\cdot)z_{iy_i}(\cdot)-w_i(w)\left(g_{iy_i}(\cdot)+r_i'(\cdot)z_{iy_i}(\cdot)\right) \\
& -\int_w^{\overline{w}}\left(1-\frac{\Phi_{u_j}(u_a(s),u_b(s))}{\lambda}\right)\pi(s)ds\frac{g_{iy_i}(\cdot)}{\pi(w)} \\
& \cdot\left(\frac{y_i(w)}{q_i'(\cdot)}\frac{\partial q_i'(\cdot)}{\partial y_i(w)}+\frac{y_i(w)}{\dot{g}_i(\cdot)}\frac{\partial \dot{g}_i(\cdot)}{\partial y_i(w)}\right)w_i'(w) \\
& -\int_w^{\overline{w}}\left(1-\frac{\Phi_{u_j}(u_a(s),u_b(s))}{\lambda}\right)\pi(s)ds\frac{r_i'(\cdot)z_{iy_i}(\cdot)}{\pi(w)} \\
& \cdot\left(\frac{y_i(w)}{q_i'(\cdot)}\frac{\partial q_i'(\cdot)}{\partial y_i(w)}+\frac{y_i(w)}{\dot{z}_i(\cdot)}\frac{\partial \dot{z}_i(\cdot)}{\partial y_i(w)}+\frac{y_i(w)}{r_i'(\cdot)}\frac{\partial r_i'(\cdot)}{\partial y_i(w)}\right)w_i'(w), \\
& \qquad i,j=a,b,\ i\neq j.
\end{aligned} \tag{197}$$

From (172), we obtain

$$\frac{dl_i}{d(1-T_i')} = -\frac{w_i}{(1+w_i g_{iy_i})\,h_i''} > 0, \quad i=a,b. \tag{198}$$

Using (174)–(176) and the definitions of elasticities, β_i, and ϱ_i, it follows that

$$\beta_i\frac{\varepsilon^i_{gT'}}{\varepsilon^i_{lT'}} = -w_i g_{iy_i}, \quad i=a,b, \tag{199}$$

$$\varrho_i\frac{\varepsilon^i_{zT'}}{\varepsilon^i_{lT'}} = w_i z_{iy_i}, \quad i=a,b. \tag{200}$$

Applying (174)–(180), (199) and (200) to (197) and utilizing the definitions of A_i^n, B_i^n, C_i^n, D_i^n, E_i^n, F_i^n, β_i, and ϱ_i yields (53).

Finally, we derive the optimal commodity tax/subsidy. Using (161), (162), (170), and (196), (190) can be rewritten as

$$\begin{aligned}
0 \;=\; & t\int_{\underline{w}}^{\overline{w}}(z_{at}(\cdot)+z_{bt}(\cdot))\pi(w)dw+\int_{\underline{w}}^{\overline{w}}(q_{at}(\cdot)+q_{bt}(\cdot))\pi(w)dw \\
& +\int_{\underline{w}}^{\overline{w}}\int_w^{\overline{w}}\left(1-\frac{\Phi_{u_a}(u_a(s),u_b(s))}{\lambda}\right)\pi(s)ds\left(-h_a''(\cdot)g_{at}(\cdot)\frac{y_a(w)}{(w_a(w))^2}+\dot{q}_{bt}(\cdot)\right)dw \\
& +\int_{\underline{w}}^{\overline{w}}\int_w^{\overline{w}}\left(1-\frac{\Phi_{u_b}(u_a(s),u_b(s))}{\lambda}\right)\pi(s)ds\left(-h_b''(\cdot)g_{bt}(\cdot)\frac{y_b(w)}{(w_b(w))^2}+\dot{q}_{at}(\cdot)\right)dw.
\end{aligned} \tag{201}$$

Using the definition of ε_{qt}, ε_{zt}, ϑ, and Ω, (201) leads to (55).