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Asset Bubbles and Aggregate Demand

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Abstract

The collapse of asset bubbles leads to a demand-driven recession. When capital utilization is endogenous and capital creation is subject to idiosyncratic risks, aggregate demand significantly influences output, even with flexible prices. The bursting of bubbles causes a sharp decline in consumption and investment demand, forcing firms to reduce capital utilization. As a result, output and long-run growth contract suddenly and severely, pushing the economy into a demand recession. Nominal rigidities further deepen the downturn. Policies that stimulate aggregate demand, such as consumption and investment subsidies, can help prevent such recessions.

Keywords: asset bubbles, demand recession, capital utilization, uninsured idiosyncratic risks, flexible price, zero lower bound.

JEL classification numbers: E32, E44, G1

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1 Introduction

Motivation and main results: The collapse of asset bubbles frequently leads to a severe recession. While booms and busts in asset prices influence economic activity in various ways, many scholars highlight the crucial role of aggregate demand. Aliber and Kindleberger (2015) provide numerous historical examples where asset bubbles triggered significant increases in both consumption and investment spending. Figure 1 (a) illustrates a sharp decline in aggregate demand during the 2007-2009 financial crisis. Empirical studies confirm that asset prices have a positive and significant effect on household consumption (see Cooper and Dynan (2016) for a survey). An increase in stock prices boosts investment demand from firms (Chirinko and Schaller (2001) and Gilchrist et al. (2005)). Conversely, asset price busts are often followed by a substantial drop in aggregate demand.

[Figure 1]

As emphasized by Keynes' effective demand theory, a significant decline in aggregate demand can cause a severe economic contraction, even without changes in production capacity. A drop in aggregate demand directly reduces firms' sales, which negatively impacts their operating rates. During the 2007-2009 financial crisis, capacity utilization across U.S. industries fell from 80 percent to 67 percent (see Figure 1 (b)).[1] Vacancy rates in office buildings also surged, surpassing 10 percent in nearly all major U.S. cities. In Los Angeles County, the office vacancy rate rose from below 10 percent in 2007 to 15.6 percent in 2009, with office rents dropping by 12.3 percent between 2008 and 2009 (USC Lusk Center, 2009). The decrease in capacity utilization and the increase in vacancy rates reduced aggregate output, driving the economy into a demand-driven recession.

This study investigates the role of aggregate demand and its interaction with asset bubbles in a growth model featuring endogenous capital utilization. The focus on aggregate demand raises several key questions: Does aggregate demand influence output even when productive capacity remains unchanged or all prices are flexible? Does the collapse of bubbles produce realistic comovement among consumption, investment, capacity utilization, the rental rate of capital, and output? Does price rigidity exacerbate the recession? Can demand-stimulating policies, such as consumption and investment subsidies, mitigate the negative effects of a bubble collapse?

This study presents four main findings. First, even with flexible prices, aggregate demand is a key determinant of output under certain conditions. The emergence of asset bubbles increases the wealth of economic agents, which in turn boosts aggregate consumption and capital formation. In response to this higher aggregate demand, firms intensify their capital utilization, which drives up the rental rate of capital. As a result, aggregate output rises, even if productive capacity (capital stock) remains unchanged.

Second, the collapse of bubbles leads to demand-driven recessions. Such crashes reduce the wealth of economic agents, resulting in an immediate contraction in both aggregate consumption and investment. Faced with reduced demand, firms cut back on capital utilization, lowering the rental rate of capital. This causes an immediate decline in production, even

[1] Authors such as Cooley and Prescott (1995) and King and Rebelo (1999) note that, in the short run, capacity utilization is far more volatile than capital stock.

though capital stock remains constant in the short run. This pattern aligns with the U.S. experience during the 2007–2009 financial crisis (see Figures 1). These contractions occur simultaneously with the bubble crash, creating a realistic comovement among consumption, investment, capacity utilization, and output in the short run. The recession persists because long-run growth is also suppressed.

Third, the zero lower bound (ZLB) on nominal interest rates worsens demand-driven recessions. Bubble crashes reduce aggregate consumption and investment demand, putting downward pressure on both output prices and interest rates. With the ZLB in place, neither can adjust fully, causing price rigidity that amplifies the effects of reduced aggregate demand. As a result, firms are forced to further reduce capital utilization, leading to significant declines in both output and long-run growth.[2]

Finally, consumption and investment subsidies are effective in combating demand-driven recessions. These policies help alleviate demand shortfalls and have a direct, positive impact on output. Furthermore, under these policies, economies without bubbles can achieve the same allocation as economies with bubbles.

Model and mechanism: In our model, all prices are flexible, eliminating the New Keynesian mechanism. Furthermore, the model excludes borrowing constraints and productivity heterogeneity, which are commonly found in recent rational bubble models. This allows us to focus on the aggregate demand channel rather than financial constraints. Instead, we introduce two key factors into a textbook growth model. First, we incorporate endogenous capital utilization, following Greenwood et al. (1988). This is motivated by the observation that capital utilization declines sharply during recessions (see Figure 1 (b)) and is much more volatile than capital stock (see footnote 1). Firms utilize capital more intensively as their output prices rise relative to input (capital) prices, generating an upward-sloping supply curve.

Second, based on empirical evidence showing that entrepreneurs face significant uninsured idiosyncratic risks (Heaton and Lucas, 2000; Moskowitz and Vissing-Jørgensen, 2002), we introduce uninsured entrepreneurial risks. Specifically, we assume that risk-averse entrepreneurs create new productive assets (capital) that are subject to uninsured risks. These entrepreneurial risks inhibit the creation of productive assets.[3] The shortage of productive assets drives up their prices and reduces their returns, which in turn stimulates demand for bubbly assets (see Caballero (2006) and Hori and Im (2023)).

With these two factors, aggregate demand becomes a crucial determinant of output. If capital utilization is exogenous, the supply curve is vertical, meaning shifts in the demand curve do not affect output (see Figure 2 (a)). However, endogenous capital utilization results in an upward-sloping supply curve (see Figure 2 (b) and (c)). In the absence of capital creation risks, the output price relative to the input (capital) price stabilizes at one (Figure 2 2 (b)). Thus, the production side entirely determines output, thereby pining down aggregate demand, even with an upward-sloping supply curve. However, when capital creation risks are present, risk-averse entrepreneurs will create capital only if the output price deviates

[2]This outcome aligns with the experiences of Japan and the U.S. Following the asset bubble bursts in the early 1990s, Japan faced historically low interest rates and a prolonged period of low growth. Similarly, after the 2007–2009 financial crisis, the U.S. struggled with low interest rates and a deep recession.

[3]Caggese (2012) and Michelacci and Schivardi (2013) demonstrate that uninsured risks disrupt entrepreneurial activities, leading to a shortage of productive assets.

from the capital price. The output price is then determined by the intersection of the supply and demand curves, meaning shifts in the aggregate demand curve have a direct impact on output (Figure 2 (c)).

[Figure 2]

An asset bubble crash reduces wealth, leading to a decline in aggregate consumption and investment, which triggers a demand-driven recession. In contrast, consumption and investment subsidies stimulate aggregate demand, shifting the demand curve to the right. This directly boosts output and mitigates the recession driven by reduced demand.

Related literature: Recent macroeconomic models have concentrated on the relationship between asset prices and economic activity. However, as noted by Caballero and Simsek (2020) and Gertler and Kiyotaki (2010), the literature has often overlooked the crucial role of aggregate demand.[4] In these models, the supply side fully determines aggregate output, a viewpoint that applies to most rational bubble models. In contrast, our approach emphasizes the role of the demand side.

Rational bubble models typically examine the effects of bubbles on productive capacities, including capital accumulation and aggregate total factor productivity (TFP) (e.g., Tirole (1985), Weil (1987), Farhi and Tirole (2011), Martin and Ventura (2012), Kunieda and Shibata (2016), Hirano and Yanagawa (2017)). Since capital adjusts slowly, the collapse of bubbles tends to have a gradual effect on output. However, Figure 1 (c) and (d) demonstrate that during the 2007–2009 financial crisis, real GDP declined sharply, while capital stock did not experience a similar downturn. Moreover, in these models, aggregate consumption and investment move in opposite directions in the short run,[5] which contradicts the observed comovement during recessions (see Figure 1 (a)). In contrast, our model focuses on how bubbles influence aggregate demand and the *utilization* of capital. We demonstrate that a bubble crash leads to an immediate and simultaneous contraction in consumption, investment, capital utilization, and output, even when capital stock remains unchanged in the short run.

Recently, several authors have emphasized the role of demand-side factors in economic booms and busts, often incorporating price rigidities and constrained interest rate policies, such as the ZLB. For example, Hanson and Phan (2017) and Biswas et al. (2020) introduce nominal wage rigidities in rational bubble models. Caballero and Simsek (2020) show that a risk premium shock can trigger a demand recession in a New Keynesian framework.[6] Basu and

[4]We discuss the literature that focuses on aggregate demand later.

[5]In Martin and Ventura (2012), equation (11) shows that bubbles in period t affect capital and output in period $t+1$. They assume heterogeneous productivity among agents. If $\delta = 1$, the right-hand side of equation (11) corresponds to aggregate investment, showing that a collapse of bubbles increases aggregate investment. In Hirano and Yanagawa (2017), equation (16) shows that the bursting of asset bubbles increases aggregate investment (though investment allocation becomes less efficient), while equation (8) shows that aggregate consumption and investment move in opposite directions. Figure 3 in Kunieda and Shibata (2016) displays similar results. In Kunieda and Shibata (2016) and Hirano and Yanagawa (2017), the collapse of bubbles affects long-run growth but has no short-run impact on output.

[6]Other examples of New Keynesian models with demand shocks causing recessions include noise shocks (Lorenzoni (2009)) and confidence shocks (Ilut and Schneider (2014)). Caballero and Simsek (2020) also argue that financial speculation during booms worsens demand recessions.

Bundick (2017) demonstrate that uncertainty shocks fail to replicate procyclical aggregate dynamics in flexible price models but succeed when nominal rigidities are introduced. Unlike these models, our framework shows that bubble crashes can induce demand recessions even without price rigidities. We do not argue that price rigidities are unimportant; in fact, they become crucial as ZLB constraints deepen demand recessions.

Di Tella and Hall (2022) recently propose a flexible price model of business cycles without TFP shocks, where increased idiosyncratic risks depress labor demand, leading to a recession. In contrast, our model assumes that the level of idiosyncratic risks remains constant over time. A bubble crash immediately reduces entrepreneurs' wealth, which lowers consumption and investment demand, forcing firms to cut capital utilization and production.

The studies mentioned above mainly focus on monetary and macroprudential policies. In contrast, both Di Tella and Hall (2022) and our study examine fiscal policies. Di Tella and Hall (2022) argue that, during a recession, lowering labor taxes and raising capital taxes are optimal policies, indirectly stimulating consumption. In contrast, we propose direct subsidies to stimulate consumption and investment.

The remainder of this paper is organized as follows. Section 2 introduces our flexible price AK model. Section 3 derives the equilibrium. Section 4 analyzes the impact of bubbles on aggregate demand and other macroeconomic indicators. Section 5 examines the dynamics following a bubble crash and demonstrates how it triggers a demand recession. Section 6 explores how the ZLB exacerbates the recession. Section 7 discusses the effectiveness of demand-stimulating policies in mitigating the downturn. Section 8 provides concluding remarks.

2 A Simple Flexible Price Model with AK Technology

Our model builds on the textbook AK framework, with time running continuously from $t = 0$ to ∞ and all prices are fully flexible. Like Greenwood *et al.* (1988), we endogenize capital utilization and introduce investment risks in the creation of productive assets.[7] However, unlike their approach, we incorporate idiosyncratic risks. Specifically, following Hori and Im (2023), risk-averse entrepreneurs face these risks when creating new capital.

Our model is similar to that of Hori and Im (2023), with the key difference being endogenous capital utilization, which is critical to the aggregate demand channel. We highlight three points. First, as in Krebs (2003) and Hori and Im (2023), introducing a risk-free bond with zero net supply (enabling borrowing and lending) does not alter the equilibrium allocation, meaning that credit constraints are not relevant (see online Appendix A.4 of Hori and Im (2023)). For simplicity, this study excludes a risk-free bond. Second, if we adopt a neoclassical production function, our main results still hold (see Appendix M). Finally, as with standard AK models, capital in our framework can be interpreted broadly. The model can be extended to a model of expanding variety (see online Appendix B of Hori and Im (2023)), where the risk of creating new capital encompasses the risks of starting new businesses, opening branches, and developing new products and technologies. Additional

[7] Guerron-Quintana et al. (2023) also explore endogenous capital utilization in a bubble model, focusing on the numerical analysis of recurrent bubbles, rather than the demand channel.

extensions concerning risks and the functional form of capital creation are noted in footnote 10. These extensions are discussed in greater detail in Hori and Im (2023).

2.1 General Good Sector

A single general good is used for both consumption and as an input in capital production. The general good is produced competitively using the following production function:

$$Y_t = A\zeta_t K_t^I, \quad A > 0, \tag{1}$$

where Y_t represents output, K_t^I denotes the capital input, and $\zeta_t \geq 0$ is the capital utilization rate. Following Greenwood *et al.* (1988), we assume that higher capital utilization leads to greater depreciation. The depreciation rate is specified as follows:

$$\delta(\zeta_t) = \delta_1 \frac{\zeta_t^{1+1/\eta}}{1+1/\eta} + \delta_2, \quad \text{where} \ \ \delta_1 > 0, \ \ \delta_2 \geq 0 \ \ \text{and} \ \ \eta > 0. \tag{2}$$

The general good is taken as the numeraire. The capital price is denoted as v_t, and the rental rate of capital is represented by q_t. The profit of the general good producer is given by:[8]

$$\Pi_t^Y = A\zeta_t K_t^I - q_t K_t^I - \delta(\zeta_t) v_t K_t^I. \tag{3}$$

The general good firm chooses ζ_t to maximize net output $(A\zeta_t - \delta(\zeta_t)v_t)K_t^I$, which yields

$$\zeta_t = \left(\frac{A}{\delta_1}\frac{1}{v_t}\right)^\eta = \left(\frac{AV_t}{\delta_1}\right)^\eta \equiv \zeta(V_t). \tag{4}$$

$V_t \equiv 1/v_t$ represents the price of the general good relative to the capital price. A higher general good price relative to the capital price encourages increased capital utilization.[9] As η increases, the production of the general good becomes more responsive to changes in its price. For simplicity, the following discussion assumes that:

$$\eta = 1.$$

As shown in (6), η influences the slope of the supply curve of the general good. The value of η determines how shifts in aggregate demand affect general good production. Appendix L demonstrates that as long as η remains strictly positive, our main results are unaffected.

Both the general good and factor markets are competitive, so we have:

$$q_t = A\zeta_t - \delta(\zeta_t)v_t, \tag{5}$$

[8] Alternatively, we can assume that the profit of the general good producer is $\Pi_t^Y = (A - Q_t)K_t^I$, with the entrepreneur receiving $q_t = Q_t\zeta_t - \delta(\zeta_t)v_t$ units of the general good for each unit of capital. The entrepreneur maximizes $q_t = Q_t\zeta_t - \delta(\zeta_t)v_t$ by choosing ζ_t. This maximization problem leads to $Q_t = A$ and $Q_t - \delta'(\zeta_t)v_t = 0$, respectively. In this case, (4), (5), and (6) still hold, and our main results remain unaffected.

[9] Caballero and Simsek (2020) assume that utilizing capital incurs no costs up to a certain threshold, and that the output price is fixed. In this framework, capital utilization is independent of the (fixed) output price. For more details, see B.1.3 in the online appendix of Caballero and Simsek (2020).

for any $\zeta_t \geq 0$. The capital input K_t^I equals the capital supply K_t. The supply of the general good is given by:

$$Y_t = A^{1+\eta}\left(\frac{V_t}{\delta_1}\right)^{\eta} K_t = \frac{A^2 V_t}{\delta_1} K_t. \tag{6}$$

Endogenous capital utilization causes the general good supply curve to slope upward with respect to V_t.

2.2 Entrepreneurs

The setup of entrepreneurs closely follows that of Hori and Im (2023). Thus, we focus on the key points. For a detailed discussion on the structure and interpretation of the model, refer to Section 2 of Hori and Im (2023).

Preferences and Asset holdings: Entrepreneurs are infinitely lived and risk-averse. The expected utility of entrepreneur $i \in [0,1]$ is given by:

$$U_{i,t} = E_t \int_t^{\infty} (\log c_{i,t})\, e^{-\rho(s-t)} ds, \tag{7}$$

where $c_{i,t}$ is the consumption of entrepreneur i, and E_t is the expectation operator conditional on information available at time t. The subjective discount rate, ρ, satisfies $A > \rho > 0$. Entrepreneur i holds $k_{i,t}$ units of productive assets (capital) and $b_{i,t}^n$ units of bubbly assets. Free disposability ensures that the price of bubbly assets is nonnegative, $p_t \geq 0$. The total assets of entrepreneur i are given by $\omega_{i,t} = v_t k_{i,t} + p_t b_{i,t}^n = a_{i,t} + b_{i,t}$, where $a_{i,t} \equiv v_t k_{i,t}$ and $b_{i,t} \equiv p_t b_{i,t}^n$. We assume that $\omega_{i,0} > 0$ for all entrepreneurs.

Entrepreneurs lend their capital to general good firms at the rental rate q_t and earn capital rental income. The rate of return on holding capital is given by:

$$r_t dt \equiv \frac{q_t dt + dv_t}{v_t}. \tag{8}$$

As in Tirole (1985), the bubbly asset is intrinsically useless and has zero fundamental value. In the *bubbleless economy*, p_t is zero (p_t=0). In the *bubbly economy*, p_t is strictly positive ($p_t > 0$). If bubbles persist, the rate of return on holding bubbly assets is:

$$\psi_t dt \equiv \frac{dp_t}{p_t}.$$

As in Weil (1987), bubbles burst stochastically due to a sunspot shock. Given that $p_t > 0$, p_{t+dt} remains strictly positive with probability $1-\mu dt$, where $\mu > 0$ is a constant. Otherwise, we have $p_{t+dt} = 0$. Once bubbles burst, they are never valued in the future.

Capital creation and Budget constraints: Capital creation is irreversible and subject to idiosyncratic risks, which are uninsurable. If entrepreneur i uses $I_{i,t}(\geq 0)$ units of the general good for a period of length dt, $dx_{i,t}$ units of new capital are produced as follows:

$$dx_{i,t} = \phi I_{i,t} dt + \sigma I_{i,t} dW_{i,t}, \qquad \phi = 1, \quad \sigma > 0, \tag{9}$$

where $W_{i,t}$ is a standard Brownian motion, and $dW_{i,t}$ represents the idiosyncratic capital creation risks, which are independently and identically distributed across entrepreneurs. A large σ indicates that entrepreneurs face significant risks.[10] As mentioned earlier, these risks encompass the risks of establishing new businesses, opening branches, and developing new products and technologies.

An aggregate market exists in which productive assets are traded.[11] Entrepreneurs sell the capital they produce at the price v_t and earn profits of $(v_t - 1)I_{i,t}dt + \sigma v_t I_{i,t} dW_{i,t}$. In the bubbly economy, entrepreneur i faces the following budget constraints (see Appendix A):

$$d\omega_{i,t} = \{[r_t(1 - s_{i,t}) + \psi_t s_{i,t}]\, \omega_{i,t} - c_{i,t}\}\, dt + I_{i,t}[(v_t - 1)dt + \sigma v_t dW_{i,t}], \tag{10}$$

where $s_{i,t} \equiv b_{i,t}/\omega_{i,t} \in [0, 1]$ is the portfolio weight on bubbly assets. In the bubbleless economy, we have $s_{i,t} = 0$ in (10).[12]

Utility maximization: At each point in time, entrepreneur i chooses $c_{i,t}$, $s_{i,t}$, and $I_{i,t}$. The households' optimization problem must satisfy the following non-negativity constraints: $k_{i,t} \geq 0$, the short sale constraint $b^n_{i,t} \geq 0$, and the no-Ponzi-game condition $\lim_{T\to\infty} \omega_{i,T} e^{-\int_t^T r_v dv} \geq 0$. We assume an interior solution for $I_{i,t} \geq 0$, which holds in the equilibrium considered. Appendix B shows that entrepreneurs' optimal behavior is summarized by:

$$c_{i,t} = \rho \omega_{i,t}, \tag{11a}$$

$$I_{i,t} = \frac{v_t - 1}{(\sigma v_t)^2} \omega_{i,t}, \quad \sigma > 0, \tag{11b}$$

$$s_{i,t} = s_t = \begin{cases} 1 - \frac{\mu}{\psi_t - r_t} > 0 & \text{in the bubbly economy } (p_t > 0), \\ 0 & \text{in the bubbleless economy } (p_t = 0), \end{cases} \tag{11c}$$

$$d\omega_{i,t} = \left[r_t(1 - s_t) + \psi_t s_t + \left(\frac{v_t - 1}{\sigma v_t} \right)^2 - \rho \right] \omega_{i,t} dt + \left(\frac{v_t - 1}{\sigma v_t} \right) \omega_{i,t} dW_{i,t}, \quad \sigma > 0. \tag{11d}$$

Since Section 2.2 in Hori and Im (2023) provides a detailed discussion of these conditions, we focus on the key points. Both consumption $c_{i,t}$ and new capital creation $I_{i,t}$ depend positively on bubbles ($\omega_{i,t} = v_t k_{i,t} + p_t b^n_{i,t}$). (11b) shows that new capital is produced ($I_{i,t} > 0$) if and only if the capital price is greater than the capital production cost ($v_t > 1$). Without uninsured risk ($\sigma = 0$), the general good price equals the capital price, $V_t(\equiv {v_t}^{-1}) = 1$. Equation (11c) determines entrepreneurs' portfolio choice between $a_{i,t}$ and $b_{i,t}$. (11d) shows that $\omega_{i,t}$ follows a generalized geometric Brownian motion, and $\omega_{i,t} > 0$ holds because of

[10] Hori and Im (2023) explore several extensions of (9), such as investment adjustment costs, heterogeneity in ϕ and σ, and aggregate risks in capital creation.

[11] The introduction of a risk-free bond in zero net supply does not change the equilibrium, as shown in Krebs (2003) and Hori and Im (2023). Thus, credit constraints are not relevant. See Proposition A1 of Section A.4 in the Online Appendix of Hori and Im (2023) for the formal proof.

[12] As in Hori and Im (2023), we distinguish between income from productive asset creation, $I_{i,t}[(v_t - 1)dt + \sigma v_t dW_{i,t}]$, and income from productive asset holdings, $r_t(1 - s_{i,t})\omega_{i,t}$. The former is subject to idiosyncratic risk, $\sigma dW_{i,t}$, while the latter bears no risk. This captures the large risks associated with establishing new businesses, opening branches, and developing new products and technologies. The introduction of risks related to asset holdings does not alter the main results. For further details, see online Appendix A.5 of Hori and Im (2023).

$\omega_{i,0} > 0$.[13] Thus, the no-Ponzi condition is satisfied. The transversality condition holds as $\lim_{t\to\infty} E_t\left[(\omega_{i,t}/c_{i,t})e^{-\rho t}\right] = \lim_{t\to\infty} \rho e^{-\rho t} = 0$.

2.3 Aggregation and Competitive Equilibrium

The following aggregate variables are defined as: $C_t = \int_0^1 c_{i,t}di$, $I_t = \int_0^1 I_{i,t}di$, $K_t = \int_0^1 k_{i,t}di$, $b_t^n = \int_0^1 b_{i,t}^n di$, and $\omega_t = \int_0^1 \omega_{i,t}di$. Then, we have:

$$\omega_t = v_t K_t + p_t b_t^n, \tag{12a}$$

$$C_t = \rho\omega_t, \tag{12b}$$

$$I_t = \frac{v_t - 1}{(\sigma v_t)^2}\omega_t, \quad \sigma > 0. \tag{12c}$$

In the equilibrium we consider, ζ_t is constant over time. Thus, the long-run growth rate of the economy is given by[14]

$$g_t = \frac{\dot{K}_t}{K_t} = \frac{I_t}{K_t} - \delta(\zeta_t). \tag{13}$$

The total nominal supply of bubbly assets is constant at $M > 0$. Thus, we have $b_t^n = M$ in bubbly asset markets. Let us define

$$B_t \equiv \frac{p_t M}{v_t K_t}. \tag{14}$$

B_t is a jump variable that represents the asset bubble size. We have $B_t = 0$ in the bubbleless economy. Since $p_t M = s_t \omega_t$ holds from $b_t = s_t \omega_t$ and $b_t^n = M$, we have $s_t = B_t/(1 + B_t)$. Thus, $s_t \in (0, 1)$ holds in the bubbly economy.

Bubbles and Demand: The general good market clears as $Y_t = C_t + I_t$. Assume that $I_t > 0$. Using (12b), (12c), and (14), we rewrite $Y_t = C_t + I_t$ as

$$\underbrace{A\zeta_t}_{Y_t/K_t} = \underbrace{\left(\frac{\rho}{V_t} + \frac{1 - V_t}{\sigma^2}\right)(1 + B_t)}_{(C_t + I_t)/K_t}, \tag{15}$$

where $\zeta_t = AV_t/\delta_1$ is the utilization rate. The left-hand side (LHS) represents the supply curve of the general good, while the right-hand side (RHS) represents the aggregate demand curve. Given other factors are constant, asset bubbles have a positive effect on aggregate demand and thus stimulate capital utilization. In the following two cases, shifts in the demand curve do not affect general good production. First, when capital utilization is exogenous. Second, as mentioned earlier, without capital creation risk ($\sigma = 0$), we have $V_t = 1$ (or $v_t = 1$), so the utilization rate is fixed at $\zeta_t = A/\delta_1$. In both cases, the supply side determines the general good output.

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[13] See Example 4.4.8 on pp.147–148 in Shreve, 2004.

[14] Because $dW_{i,t}$ has a zero mean and is independently and identically distributed among entrepreneurs, $\int_0^1 (dW_{i,t})di = 0$ holds owing to the law of large numbers (Uhlig (1996)). Since $I_{i,t}$ and $dW_{i,t}$ are independent, we aggregate (9) as $\int_0^1 (dx_{i,t})di = I_t dt + \sigma \int_0^1 I_{i,t}di \int_0^1 (dW_{i,t})di = I_t dt$. Thus, we have $dK_t \equiv [\int_0^1 (dx_{i,t})di - \delta(\zeta_t)K_t]dt = [I_t - \delta(\zeta_t)K_t]dt$.

Steady States: We consider two types of steady-state equilibria as in the rational bubble literature. A *bubbleless steady-state equilibrium* is an equilibrium in which bubbly assets have no value, V_t, ζ_t, and g_t are constant, and C_t, K_t, and Y_t grow at the same rate. A *stochastic bubbly steady-state equilibrium* is an equilibrium in which bubbly assets are valued at a positive price $B_t > 0$, V_t, ζ_t, and g_t are constant, and C_t, K_t, and Y_t grow at the same rate as long as bubbles exist. After the bubbles collapse, they are never valued. For simplicity, we refer to these equilibria as the *bubbleless steady state* and the *bubbly steady state*, respectively.

Economy without Uninsured Risks ($\sigma = 0$): If $\sigma = 0$, asset bubbles cannot exist. In this case, a unique bubbleless equilibrium exists where $V_t = 1 \equiv V_{NR}$, $\zeta_t = A/\delta_1 \equiv \zeta_{NR}$, $q_t = A\zeta_{NR} - \delta(\zeta_{NR})/V_{NR} = A^2/(2\delta_1) - \delta_2 \equiv q_{NR}$, $r_t = q_{NR}V_{NR} \equiv r_{NR}(= q_{NR})$, and $g_t = A\zeta_{NR} - \rho - \delta(\zeta_{NR}) \equiv g_{NR} \ (< r_{NR})$. The inequality $A > \rho$ ensures that $I_t > 0$. See Appendix C. With $\sigma = 0$, utilization is fixed at $\zeta_t = A/\delta_1$. Hence, the supply side entirely determines output, even if capital utilization is endogenous.

3 Uninsured Risks and Asset Bubbles

The following proposition provides a set of equations that characterize the equilibrium with uninsured risks $\sigma > 0$.

Proposition 1 *Suppose $\sigma > 0$. At an equilibrium where $I_t > 0$ holds, V_t and B_t satisfy* (15) *and*

$$\dot{B}_t = \left[\mu(1 + B_t) + A\zeta_t V_t - \frac{1 - V_t}{\sigma^2}(1 + B_t)\right] B_t, \tag{16}$$

where ζ_t is given by (4).

(Proof) See Appendix D.

With $B_t = 0$, the general good market equilibrium condition (15) reduces to:

$$\frac{A^2 V}{\delta_1} = \frac{\rho}{V} + \frac{1 - V}{\sigma^2}, \tag{17}$$

If (17) has a solution in the interval (0,1), denoted as $V_L \in (0, 1)$, the bubbleless steady state exists.

Proposition 2 *Suppose that $\sigma > 0$. If and only if $\delta_1 > 0$ is sufficiently small to satisfy*

$$\delta_1 \rho < A^2, \tag{18}$$

a unique bubbleless steady-state equilibrium exists where $I_t > 0$ holds and $V_t = V_L$, $B_t = 0$, $\zeta_t = \zeta(V_L) \equiv \zeta_L(< \zeta_{NR})$, $q_t = A\zeta_L - \delta(\zeta_L)/V_L = A^2V_L/(2\delta_1) - \delta_2/V_L \equiv q_L(< q_{NR})$, $r_t = q_L V_L = (AV_L)^2/(2\delta_1) - \delta_2 \equiv r_L(< r_{NR})$, and $g_t = (1 - V_L)/\sigma^2 - \delta(\zeta_L) \equiv g_L(< g_{NR})$. In the bubbleless economy, the only equilibrium is the bubbleless steady state.

(Proof) See Appendix E.

With uninsured risks $\sigma > 0$, entrepreneurs produce only a small amount of new productive assets (capital), leading to reduced growth ($g_L < g_{NR}$). The decreased capital production raises the capital price v_t, which in turn lowers the relative price of the general good ($V_L < V_{NR}$). As a result, capital utilization declines ($\zeta_L < \zeta_{NR}$). The higher capital price v_t also reduces the rate of return on productive assets ($r_L < r_{NR}$). This decrease in the rate of return triggers speculative demand for bubbly assets, as suggested by Caballero (2006). We now present the following proposition:

Proposition 3 *Let us define*

$$V^* \equiv 1 - \sigma(\rho+\mu)^{\frac{1}{2}} \ \in (0,\ V_{NR}) \qquad \textit{and} \qquad Z \equiv \frac{1-\sigma(\rho+\mu)^{\frac{1}{2}}}{\frac{1}{\sigma}(\rho+\mu)^{\frac{1}{2}}-\mu}. \tag{19}$$

For sufficiently large A, there exist σ_1 and σ_2 ($0 < \sigma_1 < \sigma_2 < (\rho+\mu)^{-\frac{1}{2}}$) such that for $\sigma \in (\sigma_1, \sigma_2)$, a unique bubbly steady-state equilibrium exists. In this equilibrium, the followings hold: $V_t = V^* \in (0, V_{NR})$, $\zeta_t = \zeta(V^*) \equiv \zeta^*(< \zeta_{NR})$, $B_t = ZA\zeta^* - 1 \equiv B^*(> 0)$, $q_t = A\zeta^* - \delta(\zeta^*)/V^* = A^2V^*/(2\delta_1) - \delta_2/V^* \equiv q^*(< q_{NR})$, $r_t = q^*V^* = (AV^*)^2/(2\delta_1) - \delta_2 \equiv r^*(< r_{NR})$, $\psi_t - r_t = \mu(1+B^*)(> 0)$, and $g_t = \frac{1-V^*}{\sigma^2}(1+B^*) - \delta(\zeta^*) \equiv g^*(< g_{NR})$.

(Proof) See Appendix F.

Proposition 3 demonstrates that the bubbly steady state exists if (i) technology A is sufficiently advanced and (ii) the degree of entrepreneurial risk lies within a middle range, specifically $\sigma_1 < \sigma < \sigma_2$. The intuition behind this is as follows: uninsured risk ($\sigma > \sigma_1$) reduces the rate of return on productive assets, which in turn stimulates demand for speculative bubbles. However, bubbles can only emerge if production risk is not too large ($\sigma < \sigma_2$) and A is sufficiently high, allowing capital to accumulate at a rate that supports the expansion of asset bubbles.[15]

Since a sufficiently large A satisfies (18), both the bubbly and bubbleless steady states can coexist under the same parameter set, as shown by the following corollary.

Corollary 1 *Suppose that A is sufficiently large and $\sigma \in (\sigma_1, \sigma_2)$. In this case, two steady-state equilibria exist: the bubbly steady-state equilibrium and the bubbleless steady-state equilibrium.*

4 Bubbles and Aggregate Demand

We examine how asset bubbles affect aggregate variables. Let the capital price, aggregate consumption, capital investment, and general good production in the bubbly and bubbleless steady states be denoted as v^*, C_t^*, I_t^*, and Y_t^* for the bubbly state, and v_L, $C_{L,t}$, $I_{L,t}$, and

[15]Similar to Hori and Im (2023), the mechanism behind bubble existence in our model differs from those in existing studies. In models with borrowing constraints (Martin and Ventura (2012), Kunieda and Shibata (2016), Hirano and Yanagawa (2017)), agents hold bubbles to ease borrowing constraints. Aoki *et al.* (2014) and Brunnermeier and Sannikov (2016) focused on agents holding safe bubbles to diversify asset risks. These mechanisms are not present in our model.

$Y_{L,t}$ for the bubbleless state, respectively. We omit time index t from v^* and v_L since they are constant in the steady state. In the following discussion, we use the terms capital creation and aggregate investment interchangeably. We now present the following proposition.

Proposition 4 *Suppose that both the bubbly and bubbleless steady-state equilibria exist.*
(i) Capital utilization: We have

$$V^* > V_L, \quad q^* > q_L, \quad r^* > r_L, \quad \text{and} \quad \zeta^* > \zeta_L. \tag{20a}$$

(ii) Short-run level effects: Suppose that both steady states have the same level of capital stock at time t, $K_t^* = K_{L,t}$. We have

$$C_t^* > C_{L,t} \quad \text{and} \quad Y_t^* > Y_{L,t}. \tag{20b}$$

If $\rho > 0$ is sufficiently small, we have

$$I_t^* > I_{L,t}. \tag{20c}$$

(iii) Long-run growth effects: If $\rho > 0$ is sufficiently small, we have $g^* > g_L$. In addition, if $\delta_2 \geq 0$ is sufficiently small, we have

$$g^* > g_L > 0. \tag{20d}$$

(Proof) See Appendix G.

The general equilibrium condition for the goods market, $Y_t = C_t + I_t$, or equivalently, (15), helps clarify Proposition 4 (see Figure 2 (c)). The LHS of (15) shows that the relative supply of the general good, $Y_t/K_t = A\zeta_t$, increases with the relative price V_t. A higher V_t encourages greater capital utilization, $\zeta_t \equiv AV_t/\delta_1$. On the RHS, relative aggregate demand, $(C_t + I_t)/K_t$, decreases with V_t. The intersection of these two curves (point a) determines the equilibrium value of V_L.

Asset bubbles stimulate macroeconomic activity by increasing aggregate consumption and investment, which shifts the aggregate demand curve to the right (see point b in Figure 2 (c)). This raises the relative price of the general good, $V^* > V_L$, and induces a level effect. The higher price boosts capital utilization ($\zeta^* > \zeta_L$), leading to higher production of the general good and increased capital demand. Consequently, the rental rate of capital rises ($q^* > q_L$), and the relative price of capital, $v_t (\equiv V_t^{-1})$, falls, resulting in an increased rate of return on productive assets ($r^* > r_L$). Since (20b) and (20c) hold for a given K_t, this level effect is a short-run phenomenon. Bubbles also promote long-term growth by stimulating capital production by entrepreneurs (see (11b)) and increasing general good production (see (20b)).

In our model, prices are flexible. The aggregate demand effect arises from endogenous capital utilization and entrepreneurial risk. If the capital utilization rate is fixed exogenously, the supply curve becomes vertical (see Figure 2 (a)). If $\sigma = 0$, then $V_t = 1$ and the capital utilization rate is fixed at $\zeta_t \equiv A/\delta_1$ (see Figure 2 (b)). In both cases, the supply side determines Y_t.

Bubbles and welfare: Denote the utility of entrepreneurs holding $k_{i,t}$ by $W_L(k_{i,t})$ and $W^*(k_{i,t})$ in the bubbleless and bubbly steady states, respectively. Since asset bubbles increase consumption and growth, we can easily show that bubbles enhance entrepreneurs' utility.

Proposition 5 *Suppose that Proposition 4 holds and that $\rho > 0$ is sufficiently close to zero. We have $W^*(k_{i,t}) > W_L(k_{i,t})$.*

(Proof) See Appendix H.

5 Dynamics after Bubble Crashes

We define a *demand-driven recession* as follows:

Definition 1 A *demand-driven recession* is an immediate and simultaneous contraction of aggregate consumption, aggregate investment, the capital utilization rate, output, and the growth rate, induced by a drop in aggregate demand, $C_t + I_t$ (a drop in the RHS of (15)), even though the capital stock remains constant.

To show that a bubble crash causes a demand-driven recession, we assume that Proposition 4 holds and the economy is initially in the bubbly steady state (point b in Figure 2 (c)). At $t_1 > 0$, the asset bubble collapses unexpectedly due to a sunspot shock. The economy then instantly jumps to the bubbleless steady state, which is the unique bubbleless equilibrium (see Proposition 2), because ζ_t, V_t, and B_t are jump variables. Aggregate consumption and investment drop immediately. This demand contraction deactivates some capital (with ζ_t dropping from ζ^* to ζ_L), lowers the rental rate (from q_{t_1} to q_{L,t_1}), and reduces output (from Y_{t_1} to Y_{L,t_1}) at time t_1, even though the capital stock remains constant (see Figure 3). Additionally, long-run growth slows down, resulting in a prolonged recession.

[Figure 3]

Our model addresses the limitations of existing rational bubble models, which primarily focus on supply-side effects and capital accumulation. These models fail to explain the immediate output contraction following a bubble crash, as they rely on the gradual adjustment of capital stock. As a result, they predict that aggregate consumption and investment tend to move in opposite directions, which contradicts empirical data. In contrast, our model highlights the role of demand, showing that a bubble crash can trigger an instant output contraction, leading to a realistic comovement of aggregate variables. We emphasize that these results are achieved even without nominal rigidities.

6 Nominal Price Rigidities

A bubble collapse lowers the rate of return on productive assets ($r^* > r_L$). If r_t has a lower bound, prices may not adjust fully, exacerbating a recession. To explore this possibility, we introduce a ZLB on the nominal interest rate and modify the model as follows: Denote the general good price as $P_{Y,t}$. $P_{Y,t}$ grows at rate ε_t, with v_t^n, p_t^n, and q_t^n representing the nominal prices of productive assets, bubbly assets, and the nominal rental rate of capital, respectively. We define $v_t = v_t^n/P_{Y,t}$, $p_t = p_t^n/P_{Y,t}$, $q_t = q_t^n/P_{Y,t}$, and $V_t = 1/v_t = P_{Y,t}/v_t^n$. Without the ZLB, the previous results remain unchanged.

As in Benhabib et al. (2001), we consider a simplified Taylor rule in which the monetary authorities set the nominal interest rate as a non-decreasing function of inflation, $r^n(\varepsilon)$:

$$r^n(\varepsilon) = \begin{cases} R(\varepsilon) > 0 & \text{if } \varepsilon > \hat{\varepsilon}, \\ 0 & \text{if } \varepsilon \leq \hat{\varepsilon}, \end{cases} \tag{21}$$

where $R'(\varepsilon) > 1$, $R(\hat{\varepsilon}) = 0$, and $\hat{\varepsilon}$ is a parameter. For high inflation, the monetary authorities follow an active monetary policy, $R'(\varepsilon) > 1$.[16] Given the real interest rate r_t, the inflation rate ε_t is endogenously determined by the Fisher equation: $r_t = r_t^n(\varepsilon_t) - \varepsilon_t$. Under (21), $r_t \geq -\hat{\varepsilon}$ holds in equilibrium. If $r_t = -\hat{\varepsilon}$, the inflation rate is uniquely determined, $\varepsilon = \hat{\varepsilon}$. If $r_t > -\hat{\varepsilon}$, the inflation rate is indeterminate. We assume that if $r_t > -\hat{\varepsilon}$, the highest inflation rate satisfying the Fisher equation prevails, meaning $\varepsilon_t > \hat{\varepsilon}$ when $r_t > -\hat{\varepsilon}$.

[Figure 4]

Without the ZLB, we have $r_t = (AV_t)^2/(2\delta_1) - \delta_2$ in both the bubbly and the bubbleless steady states (see Propositions 2 and 3). Given $\hat{\varepsilon} < \delta_2$, the condition $r_t (\equiv (AV_t)^2/(2\delta_1) - \delta_2) \geq -\hat{\varepsilon}$ implies:

$$V_t \geq \frac{\sqrt{2\delta_1(\delta_2 - \hat{\varepsilon})}}{A} \equiv \hat{V}. \tag{22}$$

If $V_L > \hat{V}$ and $V^* > \hat{V}$ hold, we have $r_t > -\hat{\varepsilon}$, and all the results obtained so far remain unaffected (see Figure 4 (a)), where V_L and V^* are defined in Section 3. However, if $V_L \leq \hat{V}$ or $V^* \leq \hat{V}$ holds, Proposition 2 or 3 no longer holds. Inequality (22) acts as downward rigidity on the relative price of the general good. If the general good firm sets $\zeta_t = AV_t/\delta_1$ according to (4), an excess supply of the general good occurs (Figure 4 (b)). As a result, the utilization rate becomes smaller than $\zeta_t = AV_t/\delta_1$ and is determined by aggregate demand:

$$A\zeta = \left(\frac{\rho}{V} + \frac{1-V}{\sigma^2} \right)(1+B), \qquad \zeta < AV/\delta_1. \tag{23}$$

Here, we omit the time index t. If $\zeta < AV/\delta_1$, *underutilization* occurs. With underutilization, the ZLB binds, $r^n = 0$ ($\varepsilon = \hat{\varepsilon}$). Hence, V_t and ζ_t must satisfy

$$(r^n \equiv) A\zeta V - \delta(\zeta) + \hat{\varepsilon} = 0. \tag{24}$$

In an equilibrium with underutilization, (23) and (24) are satisfied. We prove the following proposition.

Proposition 6 *Suppose that without the ZLB, both the bubbly and the bubbleless steady-state equilibria exist.*
(i) If $\hat{\varepsilon} > \delta_2$, underutilization does not occur in either the bubbly or the bubbleless steady states.
(ii) If $\underline{\varepsilon} < \hat{\varepsilon} < \delta_2 - \rho/2$ and A is sufficiently large, a unique bubbleless steady-state equilibrium with underutilization exists. Here, $\underline{\varepsilon}$ is defined in Appendix I.
(iii) Even when $\hat{\varepsilon} < \delta_2$, if A is sufficiently large, a unique bubbly steady state without underutilization exists. The steady state is the same as that in Proposition 3.

[16] Caballero and Simsek (2020) assume that prices are perfectly rigid, meaning the inflation rate is zero, instead of assuming a simplified Taylor rule. Even if we adopt their assumption, our result remains unaffected.

(Proof) See Appendix I.

If the inflation rate is high ($\hat{\varepsilon} > \delta_2$), underutilization never occurs (Proposition 6 (i)). It occurs only if the inflation rate is not large enough.[17] If A is large enough, underutilization does not occur in the bubbly economy, while it is more likely in the bubbleless economy. High productivity (large A) shifts the supply curve of the general good rightward, reducing V_L, which makes underutilization more likely in the bubbleless economy. In the bubbly economy, productivity A also affects aggregate demand. Proposition 3 shows that an increase in A expands bubbles ($B^* = ZA^2V^*/\delta_1 - 1$) and increases aggregate demand. Thus, with a large A, underutilization does not occur in the bubbly steady state.

Proposition 6 suggests that a bubble crash is likely to induce underutilization if underutilization was absent before the crash. Assume the economy is initially at a bubbly steady state without underutilization, implying a large A (see Proposition 6 (iii)). If the bubble suddenly collapses, the large A leads to underutilization according to Proposition 6 (ii).

The following proposition examines the impacts of the ZLB. We use the subscript U for the variables in the bubbleless steady state with underutilization (V_U, ζ_U, $C_{U,t}$, $I_{U,t}$, $Y_{U,t}$, and g_U) and omit the time index t from the variables that are constant over time.

Proposition 7 *Suppose that a unique bubbleless steady-state equilibrium exists, regardless of the presence of the ZLB. Further, suppose that if the ZLB is present, underutilization occurs in the bubbleless steady-state equilibrium. Then, we have:*

$$V_U > V_L, \quad \zeta_U < \zeta_L, \quad \frac{C_{U,t}}{K_t} < \frac{C_{L,t}}{K_t}, \quad \frac{I_{U,t}}{K_t} < \frac{I_{L,t}}{K_t}, \quad \text{and} \quad \frac{Y_{U,t}}{K_t} < \frac{Y_{L,t}}{K_t}.$$

In addition, if $\rho > 0$ is sufficiently small, we have

$$g_U < g_L.$$

(Proof) See Appendix J.

Underutilization exacerbates the demand-driven recession. The ZLB acts as downward rigidity, ensuring that $V_U > V_L$. The higher general good price depresses aggregate consumption and investment, leading to an excess supply of the general good. As a result, capital utilization must decrease, which causes a reduction in general good production. This, in turn, severely discourages long-run growth.

7 Aggregate Demand Policy

This section considers a policy designed to stimulate aggregate demand C_t and I_t. The budget constraint of entrepreneur i is modified as follows:

$$d\omega_{i,t} = \{(1-\tau_{\omega,t})\left[r_t(1-s_{i,t}) + \psi_t s_{i,t}\right]\omega_{i,t} - (1-\tau_c)c_{i,t}\}\,dt + I_{i,t}[(1+\tau_I)(v_t-1)dt + \sigma v_t dW_{i,t}], \tag{25}$$

where $\tau_c \in [0, 1)$ and $\tau_I > 0$ are the subsidy rates on consumption and capital creation, respectively. These policy instruments are constant over time. $\tau_{\omega,t}$ represents the tax rate

[17] If $\hat{\varepsilon} < \underline{\varepsilon}$, the ZLB is never satisfied because $r_t < -\hat{\varepsilon}$ holds for any $V \in (0, 1)$. Thus, no equilibrium exists.

on asset income. The subsidies on consumption and capital creation are financed by the asset income tax: $\tau_c C_t + \tau_I(v_t - 1)I_t = \tau_{\omega,t}\left[r_t(1-s_t) + \psi_t s_t\right]\omega_t$. The utility maximization of entrepreneur i yields:

$$c_{i,t} = \frac{\rho}{1-\tau_c}\omega_{i,t}, \tag{26a}$$

$$I_{i,t} = \frac{(1+\tau_I)(v_t - 1)}{(\sigma v_t)^2}\omega_{i,t}, \tag{26b}$$

$$s_{i,t} = s_t = \begin{cases} 1 - \frac{\mu}{(1-\tau_{\omega,t})(\psi_t - r_t)} & \text{in the bubbly economy } (p_t > 0), \\ 0 & \text{in the bubbleless economy } (p_t = 0). \end{cases} \tag{26c}$$

The subsidies τ_c and τ_I have positive effects on consumption and investment, respectively. The asset income tax $\tau_{\omega,t}$ discourages holding bubbly assets.

The general good market equilibrium condition, $Y_t = C_t + I_t$, is given by:

$$A\zeta_t = \left[\frac{1}{(1-\tau_c)}\frac{\rho}{V_t} + (1+\tau_I)\frac{1-V_t}{\sigma^2}\right](1+B_t), \tag{27}$$

where $\zeta_t = AV_t/\delta_1$ is the utilization rate. Again, the LHS and RHS of (27) are the supply of and demand for the general good, respectively. The dynamics of B_t are governed by:

$$\dot{B}_t = \left[\frac{\mu}{1-\tau_{\omega,t}}(1+B_t) + A\zeta_t V_t - (1+\tau_I)\frac{1-V_t}{\sigma^2}(1+B_t)\right]B_t. \tag{28}$$

We denote the variables in the bubbleless steady-state equilibrium under this policy as V_L^τ, ζ_L^τ, $C_{L,t}^\tau$, $I_{t,\tau}^\tau$, $Y_{L,t}^\tau$, $r_{L,t}^\tau$, and g_L^τ.

Both the consumption and investment subsidies directly affect output. Consider the bubbleless steady state. With $B_t = 0$, (27) determines V_L^τ. An increase in τ_c at time t stimulates aggregate consumption, positively affecting the RHS of (27). As a result, general good firms increase capital utilization and output at time t. The consumption subsidy has an immediate and direct impact on output. τ_I has similar effects. Thus, the demand-stimulating policy may eliminate underutilization, as the following proposition shows:

Proposition 8 *Suppose that when $\tau_c = \tau_I = 0$, both bubble and bubbleless steady-state equilibria exist. Further, suppose that no underutilization occurs in the bubbly steady state, but underutilization occurs in the bubbleless steady state. If τ_c and τ_I satisfy*

$$\frac{1}{1-\tau_c} = 1 + \tau_I = 1 + B^*, \tag{29}$$

the following statements hold. (Here, B^* is characterized in Proposition 3.). Then, a unique bubbleless steady-state equilibrium exists without underutilization. The allocation and prices in this equilibrium are the same as those in the bubbly steady-state equilibrium when $\tau_c = \tau_I = 0$:

$$V_L^\tau = V^*, \quad r_L^\tau = r^*, \quad \zeta_L^\tau = \zeta^*, \quad \frac{C_{L,t}^\tau}{K_t} = \frac{C_t^*}{K_t}, \quad \frac{I_{L,t}^\tau}{K_t} = \frac{I_t^*}{K_t}, \quad \frac{Y_{L,t}^\tau}{K_t} = \frac{Y_t^*}{K_t}, \quad \textit{and} \quad g_L^\tau = g^*,$$

where V^*, r^*, ζ^*, C_t^*, I_t^*, Y_t^*, and g^* are characterized in Proposition 3.

(Proof) See Appendix K.

The demand-stimulating policy shifts the demand curve to the right, producing effects similar to those of asset bubbles. As a result, with this policy, the bubbleless economy achieves the same allocation as the bubbly economy. If no underutilization occurs in the bubbly economy, underutilization does not occur in the bubbleless economy either.

8 Discussion and Conclusion

We develope a simple model of rational bubbles with an infinitely lived agent, where aggregate demand plays a key role in determining output. Even with flexible prices, the bursting of bubbles leads to a demand-driven recession, characterized by a sharp economic contraction and prolonged stagnation. The mechanisms behind this type of recession differ from those in New Keynesian models. In our model, endogenous capital utilization and idiosyncratic risks are critical factors. We find that nominal rigidities, such as interest rate constraints that lead to underutilized capital, worsen demand recessions. Stimulating aggregate consumption and investment through policy can help mitigate such recessions.

However, our model has several limitations. First, many credit booms end in economic crises, and the interaction between bubbles and credit booms may intensify demand recessions. Incorporating credit frictions into our model would be a valuable extension. Second, recessions often lead to job losses, but since our model excludes labor, it may underestimate the negative impact of demand recessions on employment. A more comprehensive model that includes unemployment is needed. Lastly, this study is purely theoretical, and exploring the quantitative implications of demand recessions triggered by bubble collapses would be important. Given our focus on aggregate demand, our model offers new insights into these issues.

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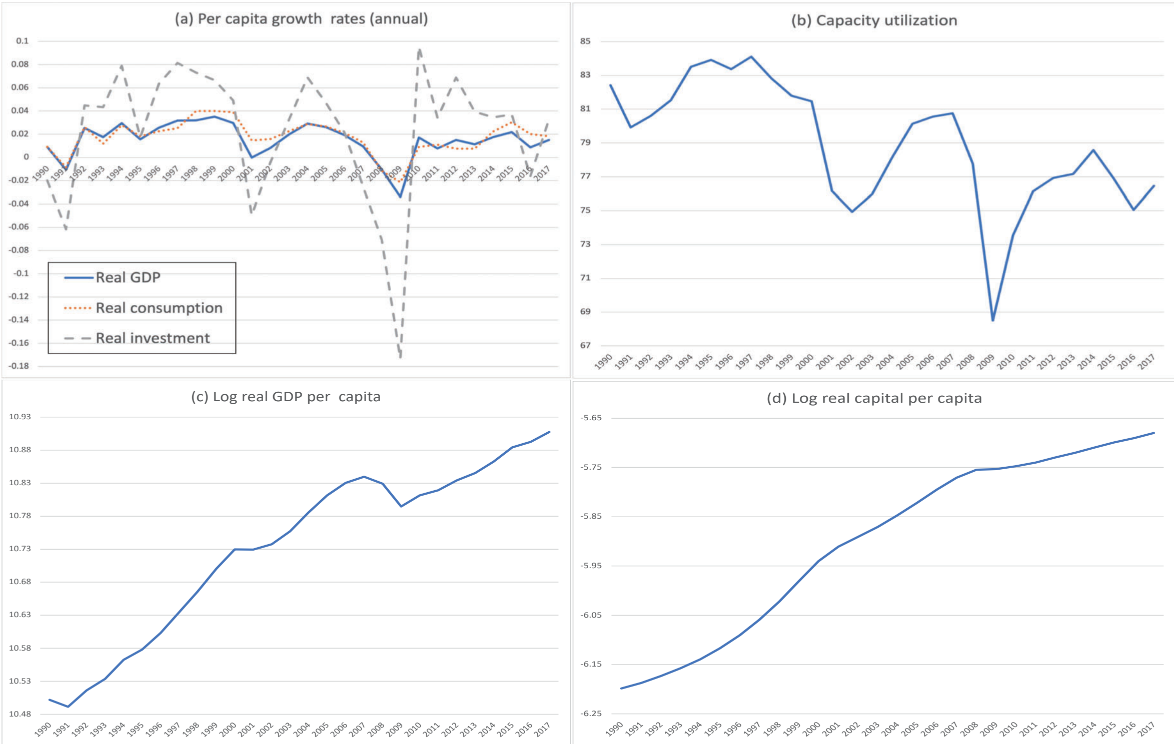


Figure 1

Source: Capacity utilization data are sourced from the Board of Governors of the Federal Reserve System (G.17 Industrial Production and Capacity Utilization). GDP and population data come from Penn World Table 9.1 (Feenstra et al. (2015)). Consumption and investment data are also taken from the National Accounts in Penn World Table 9.1. Capital data are obtained from "rkna" series in Penn World Table 9.1. The capital measure is adjusted for differences in marginal products across various capital types, making it an appropriate measure of capital input. All data are annual.

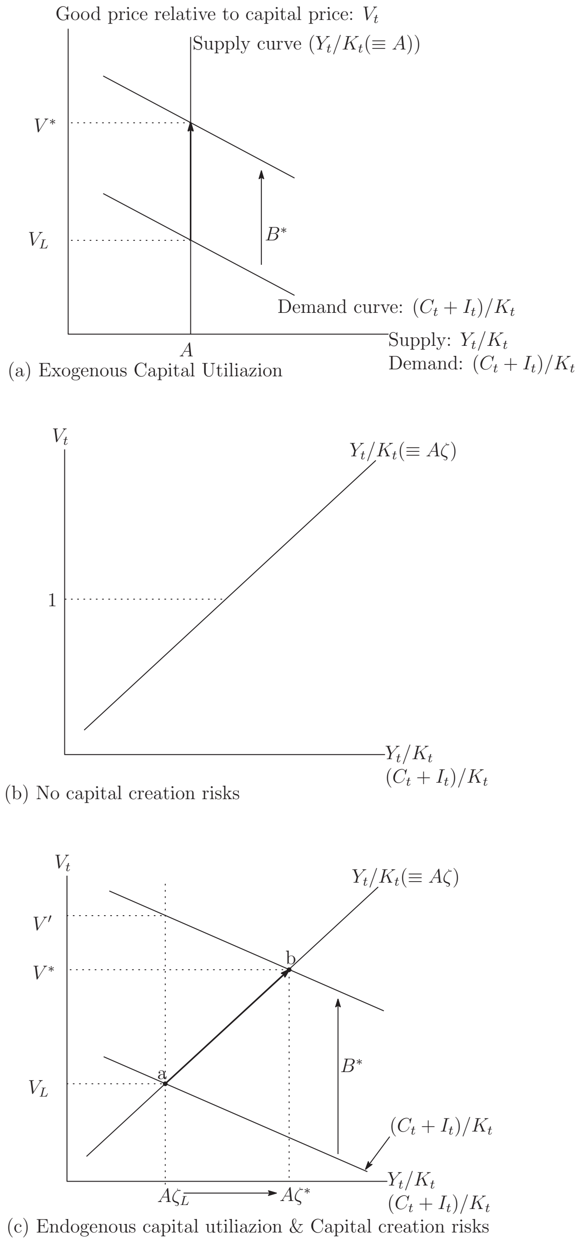


Figure 2 General Good Market and Bubbles

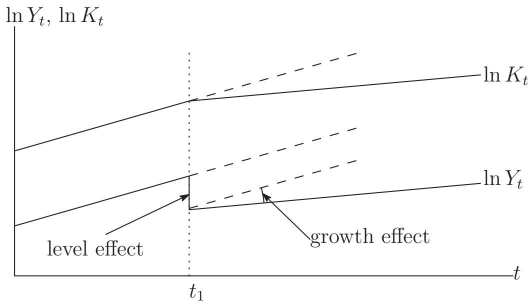
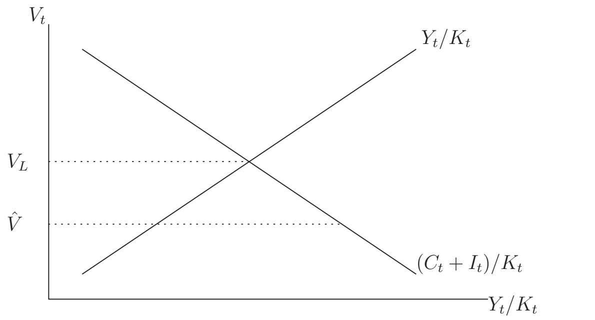
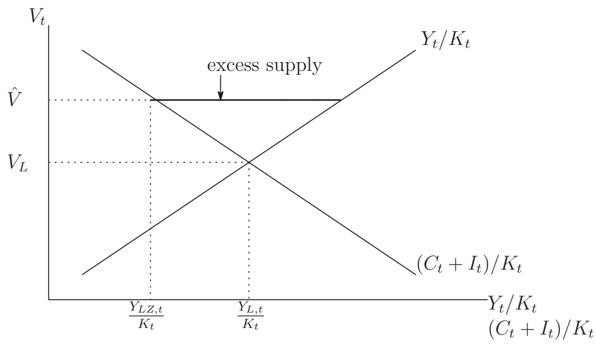


Figure 3 Collapse of Bubbles



(a) non-binding zero lower bound



(b) binding zero lower bound

Figure 4 General Good Market and ZLB

Appendix

A Derivation of the Budget Constraint (10)

Suppose a bubbly economy prevails between time t and $t + dt$. During this period, entrepreneur i earns capital rental income of $q_t k_{i,t} dt$ and profits given by $(v_t - 1)I_{i,t}dt + \sigma v_t I_{i,t} dW_{i,t}$. They consume $c_{i,t}dt$ units of the general good and purchase $dk_{i,t}$ units of capital and $db^n_{i,t}$ units of bubbly assets. If they sell capital $(dk_{i,t})$ or bubbly assets $(db^n_{i,t})$, these quantities will be negative. Thus, we have:

$$c_{i,t}dt + v_t dk_{i,t} + p_t db^n_{i,t} = q_t k_{i,t} dt + (v_t - 1)I_{i,t}dt + \sigma v_t I_{i,t} dW_{i,t}. \tag{A.1}$$

From $\omega_{i,t} = v_t k_{i,t} + p_t b^n_{i,t} = a_{i,t} + b_{i,t}$, we have $d\omega_{i,t} = (dv_t)k_{i,t} + v_t dk_{i,t} + (dp_t)b^n_{i,t} + p_t db^n_{i,t}$. Using $b_{i,t} = s_{i,t}\omega_{i,t}$, $a_{i,t} = (1 - s_{i,t})\omega_{i,t}$, $\omega_{i,t} = a_{i,t} + b_{i,t}$, and (A.1), the budget constraint (10) is derived.

B Bellman Equation and the Optimal Behavior of an Entrepreneur

This appendix follows Appendix B of Hori and Im (2023). In the bubbly economy, let the value function of entrepreneur i be denoted as $U^*(\omega_{i,t}, t)$, where $\omega_{i,t}$ represents the wealth of entrepreneur i. In the bubbleless economy, we set $\omega_{i,t} = a_{i,t}$, so $U(a_{i,t}, t)$ is the value function for the bubbleless economy. We assume the functional form $U(a_{i,t}, t) = D(\ln a_{i,t} + u_t)$. Equation (B.6) in Appendix B of Hori and Im (2023) shows that the Bellman equation in the bubbleless economy is given by:

$$\rho U(a_{i,t}, t) = \max_{c_{i,t}, I_{i,t}} \left\{ \log c_{i,t} + D\frac{r_t a_{i,t} + (v_t - 1)I_{i,t} - c_{i,t}}{a_{i,t}} - \frac{D}{2}\left(\frac{\sigma v_t I_{i,t}}{a_{i,t}}\right)^2 + D\dot{u}_t \right\}, \tag{B.1}$$

where D is an undetermined coefficient. The first-order conditions are given by:

$$c_{i,t}: \quad \frac{1}{c_{i,t}} = \frac{D}{a_{i,t}}, \tag{B.2}$$

$$I_{i,t}: \quad \frac{v_t - 1}{a_{i,t}} = \left(\frac{\sigma v_t}{a_{i,t}}\right)^2 I_{i,t}. \tag{B.3}$$

If we use (B.1), (B.2), and (B.3), we obtain:

$$\rho D \log a_{i,t} + \rho D u_t = \log a_{i,t} - \log D + D\left[r_t + \left(\frac{v_t - 1}{\sigma v_t}\right)^2\right] - 1 - \frac{D}{2}\left(\frac{v_t - 1}{\sigma v_t}\right)^2 + D\dot{u}_t. \tag{B.4}$$

Then, we obtain:

$$D = \frac{1}{\rho} \tag{B.5}$$

$$\rho u_t = \rho \ln \rho + r_t + \left(\frac{v_t - 1}{\sigma v_t}\right)^2 - \rho - \frac{1}{2}\left(\frac{v_t - 1}{\sigma v_t}\right)^2 + \dot{u}_t. \tag{B.6}$$

Then, we have:

$$c_{i,t} = \rho a_{i,t}.$$

The transversality condition is satisfied as $\lim_{t\to\infty} E_t\left[\frac{a_{i,t}}{c_{i,t}}e^{-\rho t}\right] = \lim_{t\to\infty}\rho e^{-\rho t} = 0$.

To consider the Bellman equation in the bubbly economy, we distinguish the capital price in the bubbly economy, denoted as v_t^* from the capital price in the bubbleless economy, denoted as v_t, because the existence of bubbles may influence the capital price. Equation (B.12) in Appendix B of Hori and Im (2023) shows that the Bellman equation in the bubbly economy can be written as:

$$\begin{aligned}\rho U^*(\omega_{i,t}, t) = \max_{c_{i,t}, I_{i,t}, s_{i,t}} \Bigg\{ & \log c_{i,t} + D^* \frac{\left[r_t(1-s_{i,t}) + \psi_t s_{i,t}\right]\omega_{i,t} + (v_t^* - 1)I_{i,t} - c_{i,t}}{\omega_{i,t}} \\ & - \frac{D^*}{2}\left(\frac{\sigma v_t^* I_{i,t}}{\omega_{i,t}}\right)^2 + D^*\dot{u}_t^* \\ & - \mu\left[D^*\left(\log\omega_{i,t} + u_t^*\right) - D\left(\log\frac{v_t}{v_t^*}(1-s_{i,t})\omega_{i,t} + u_t\right)\right]\Bigg\}. \end{aligned} \tag{B.7}$$

D^* is an undetermined coefficient. The third line uses $a_{i,t} = v_t k_{i,t} = v_t(\omega_{i,t} - b_{i,t})/v_t^* = v_t(1-s_{i,t})\omega_{i,t}/v_t^*$. The first-order conditions are given by

$$c_{i,t}: \quad \frac{1}{c_{i,t}} = \frac{D^*}{\omega_{i,t}} \tag{B.8}$$

$$I_{i,t}: \quad \frac{v_t^* - 1}{\omega_{i,t}} = \left(\frac{\sigma v_t^*}{\omega_{i,t}}\right)^2 I_{i,t} \tag{B.9}$$

$$s_{i,t}: \quad D^*(\psi_t - r_t) = D\frac{\mu}{1 - s_{i,t}}. \tag{B.10}$$

From (B.10), we obtain:

$$s_{i,t} = 1 - \frac{D}{D^*}\frac{\mu}{\psi_t - r_t} = s_t. \tag{B.11}$$

Thus, all entrepreneurs hold the same proportion of their wealth as bubbly assets.

Using (B.8), (B.9), (B.10), and (B.11), we rewrite (B.7) as:

$$\begin{aligned}\rho D^*\log\omega_{i,t} + \rho D^* u_t^* = & \log\omega_{i,t} - \log D^* + D^*\left[r_t(1-s_t) + \psi_t s_t + \left(\frac{v_t^*-1}{\sigma v_t^*}\right)^2\right] - 1 + D^*\dot{u}_t^* \\ & - \frac{D^*}{2}\left(\frac{v_t^*-1}{\sigma v_t^*}\right)^2 - \mu\left[D^*(\ln\omega_{i,t} + u_t^*) - D\left(\ln\frac{v_t}{v_t^*}(1-s_t)\omega_{i,t} + u_t\right)\right]. \end{aligned} \tag{B.12}$$

Then, we obtain:

$$D^* = \frac{1}{\rho}(= D) \tag{B.13}$$

$$\rho u_t^* = \rho \ln \rho + r_t(1-s_t) + \psi_t s_t + \left(\frac{v_t^*-1}{\sigma v_t^*}\right)^2 - \rho - \frac{1}{2}\left(\frac{v_t^*-1}{\sigma v_t^*}\right)^2$$
$$+ \mu \left\{ \ln\left[(1-s_t)\frac{v_t^*}{v_t}\right] - u_t^* + u_t \right\} + \dot{u}_t^*. \tag{B.14}$$

The behavior of entrepreneur i is summarized by (11a)–(11c), and the transversality condition holds as $\lim_{t\to\infty} E_t\left[\frac{\omega_{i,t}}{c_{i,t}} e^{-\rho t}\right] = \lim_{t\to\infty} \frac{1}{\rho} e^{-\rho t} = 0$. For simplicity, we do not distinguish between v_t^* and v_t. Substituting (11a)–(11c) into (10) yields (11d). This equation shows that $\omega_{i,t}$ follows a generalized geometric Brownian motion, ensuring that $\omega_{i,t} > 0$ because of the initial condition $\omega_{i,0} > 0$ (see Example 4.4.8 on pp. 147–148 in Shreve, 2004).

C Equilibrium without Risks $\sigma = 0$

Assume no risks are associated with capital production, that is, $\sigma = 0$. From the first-order condition for $I_{i,t}$, either (B.3) or (B.9), we obtain $V_t = 1 \equiv V_{NR}$. Hence, the capital price v_t remains constant at $1 (= \phi^{-1})$ and $\dot{v}_t = 0$. The capital utilization rate is given by $\zeta_t = A/\delta_1 \equiv \zeta_{NR}$. The rental rate of capital is $q_t = A\zeta_{NR} - \delta(\zeta_{NR})/V_{NR} = A^2/(2\delta_1) - \delta_2 \equiv q_{NR}$. The rate of return on holding capital is $r_t = q_{NR}V_{NR} = A^2/(2\delta_1) - \delta_2 \equiv r_{NR}$.

Next, we show that $B_t = 0$. Using the definition $B_t = p_t M/(v_t K_t)$, $V_t = 1 \equiv V_{NR}$, and (12b), the good market clearing condition (15) can be rewritten as:

$$A = \rho(1 + B_t) + \frac{I_t}{K_t}. \tag{C.1}$$

Since $I_t \geq 0$, $B_t = p_t M/(v_t K_t) \geq 0$ must be bounded above. Suppose that the price of bubbly assets is positive, $p_t > 0$. Then, we have:

$$\dot{B}_t = \left(\psi_t - \frac{\dot{K}_t}{K_t}\right) B_t = \{\psi_t - A + \rho(1+B_t) + \delta(\zeta_{NR})\} B_t = \{\psi_t - r_t + \rho(1+B_t)\} B_t$$
$$= (\mu + \rho)(1 + B_t) B_t. \tag{C.2}$$

The first equality uses $v_t = 1$, $\dot{v}_t = 0$, and $\psi \equiv \dot{p}_t/p_t$. The second equality uses (13) and (C.1). The third equality uses $v_t = 1$, $dv_t = 0$, and $r_t \equiv \frac{q+\dot{v}_t}{v_t}$. The last equality uses $v_t = 1$, $a_t = (1-s_t)\omega_t$, (11c), and:

$$\frac{\mu}{\psi_t - r_t} = 1 - s_t = \frac{v_t K_t}{v_t K_t + p_t M} = \frac{1}{1 + B_t}. \tag{C.3}$$

Since $B_t \geq 0$ must be bounded, the solution of (C.2) is $B_t = 0$. Thus, no bubble equilibrium exists. From $B_t = 0$ and (C.1), we have $I_t/K_t = A - \rho > 0$. From (13) and (C.1), we obtain $g_t = A\zeta_{NR} - \rho - \delta(\zeta_{NR}) \equiv g_{NR}$ $(< r_{NR})$.

D Proof of Proposition 1

If $I_t > 0$, then (12c) holds. In equilibrium, the general good market must clear ($Y_t = C_t + I_t$). Thus, (15) must hold. In the bubbly economy $B_t > 0$, we can derive the dynamics of B_t as follows:

$$\frac{\dot{B}_t}{B_t}=\frac{\dot{p}_t}{p_t}-\frac{\dot{v}_t}{v_t}-\frac{\dot{K}_t}{K_t}=\mu(1+B_t)+A\zeta_t V_t-\frac{1-V_t}{\sigma^2}(1+B_t).$$

In the second equality, we use equations (5), (12c), (13), and (C.3), where $r_t \equiv \frac{q+\dot{v}_t}{v_t}$, and $\psi_t \equiv \dot{p}_t/p_t$. In the bubbleless economy, we have $p_t = 0$, which implies that $B_t = \dot{B}_t = 0$. Then, (16) holds in both the bubbly and the bubbleless economies.

E Proof of Proposition 2

In the bubbleless economy ($B_t = 0$), (17) characterizes the equilibria. The LHS of (17) increases from 0 to A^2/δ_1 as V increases from 0 to 1. The RHS of (17) decreases from $+\infty$ to ρ as V increases from 0 to 1. Under the condition given by (18), (17) has a unique positive solution, denoted as $V_L \in (0,\ 1)$. Thus, we have $V_t = V_L$. Since $V_L < 1$, $I_t > 0$. As V_L is constant over time, this represents a steady-state equilibrium. Given the uniqueness of V_L, no other equilibria exist.

As (4) holds, we have:

$$\zeta_t=\frac{AV_L}{\delta_1}=\zeta(V_L)\equiv\zeta_L. \tag{E.1}$$

Because of $V_L < 1 = V_{NR}$, we have $\zeta_L < \zeta_{NR}$. From (5), we have:

$$q_t=A\zeta_L-\delta(\zeta_L)/V_L=\frac{A^2V_L}{2\delta_1}-\frac{\delta_2}{V_L}\equiv q_L. \tag{E.2}$$

From $r_t dt = (q_t dt + dv_t)/v_t$ and $dv_t = 0$, we have:

$$r_t=A\zeta_L V_L-\delta(\zeta_L)=\frac{(AV_L)^2}{2\delta_1}-\delta_2\equiv r_L. \tag{E.3}$$

Both q_L and r_L increase with V. Since $V_L < V_{NR}$, we have $q_L < q_{NR}$ and $r_L < r_{NR}$. Because the growth rate is given by (13), we have:

$$g_t=\frac{1-V_L}{\sigma^2}-\delta(\zeta_L)\equiv g_L. \tag{E.4}$$

From the definition of g_{NR}, (17), and (E.4), we have:

$$g=\frac{A^2V}{\delta_1}-\frac{\rho}{V}-\frac{(AV)^2}{2\delta_1}-\delta_2, \tag{E.5}$$

where $(g, V) = (g_{NR}, V_{NR})$ or $(g_{_L}, V_L)$. This equation shows that g increases with V as long as $0 < V \leq 1$. Since $0 < V_L < V_{NR}$, we have $g_L < g_{NR}$.

F Proof of Proposition 3

If we set $\dot{B}_t = 0$ in (16), we obtain:

$$\left(\frac{1 - V_t}{\sigma^2} - \mu\right)(1 + B_t) = A\zeta_t V_t. \tag{F.1}$$

If we eliminate B_t from (15) and (F.1), we obtain $V_t = 1 - \sigma(\rho + \mu)^{\frac{1}{2}} \equiv V^*$. The inequality $0 < \sigma < (\rho + \mu)^{-\frac{1}{2}}$ ensures that $V^* \in (0, 1)$. Since $V^* < 1$, we have $I_t > 0$. Because (4) holds, we have:

$$\zeta_t = \frac{AV^*}{\delta_1} \equiv \zeta^* \quad (< \zeta_{NR}). \tag{F.2}$$

As $V^* < 1 = V_{NR}$, we have $\zeta^* < \zeta_{NR}$. If we substitute $V_t = V^*$ into (F.1), we obtain $B_t = ZA\zeta^* - 1 \equiv B^*$. If $\sigma = 0$, we have $Z = 0$. In addition, we have:

$$\frac{1}{(\rho + \mu)^{\frac{1}{2}}} < \frac{1}{(\rho + \mu)^{\frac{1}{2}}} \frac{\rho + \mu}{\mu} = \frac{(\rho + \mu)^{\frac{1}{2}}}{\mu}. \tag{F.3}$$

Thus, the inequality $0 < \sigma < (\rho + \mu)^{-\frac{1}{2}}$ ensures that $Z > 0$. The inequality $B^* > 0$ holds if and only if:

$$(\rho + \mu)^{\frac{1}{2}} - \mu\sigma < A^2\sigma \left[1 - \sigma(\rho + \mu)^{\frac{1}{2}}\right]^2. \tag{F.4}$$

The LHS decreases with σ and is positive for $\sigma \in [0, 1/(\rho + \mu)^{\frac{1}{2}}]$ because of (F.3). The RHS is equal to zero if $\sigma = 0$ or $\sigma = (\rho + \mu)^{-\frac{1}{2}}$ and is positive if $\sigma \in (0, (\rho + \mu)^{-\frac{1}{2}})$. We have:

$$\frac{\partial RHS}{\partial \sigma} = A^2 \left[1 - \sigma(\rho + \mu)^{\frac{1}{2}}\right] \left[1 - 3\sigma(\rho + \mu)^{\frac{1}{2}}\right]. \tag{F.5}$$

The RHS increases with σ for $\sigma \in [0, (\rho + \mu)^{-\frac{1}{2}}/3]$ and decreases with σ for $\sigma \in [(\rho + \mu)^{-\frac{1}{2}}/3, (\rho + \mu)^{\frac{-1}{2}}]$. As A increases, the RHS also increases if $\sigma \in (0, (\rho + \mu)^{-\frac{1}{2}})$. Thus, for sufficiently large A, there exist σ_1 and σ_2 ($0 < \sigma_1 < \sigma_2 < (\rho + \mu)^{-\frac{1}{2}}$) such that (F.4) holds if $\sigma \in (\sigma_1, \sigma_2)$.

From (5), we have:

$$q_t = A\zeta^* - \frac{\delta(\zeta^*)}{V^*} = \frac{A^2 V^*}{2\delta_1} - \frac{\delta_2}{V^*} \equiv q^*. \tag{F.6}$$

From $r_t dt = (q_t dt + dv_t)/v_t$ and $dv_t = 0$, we have

$$r_t = A\zeta^* V^* - \delta(\zeta^*) = \frac{(AV^*)^2}{2\delta_1} - \delta_2 \equiv r^* \quad (< r_{NR}). \tag{F.7}$$

Since $V^* < 1 = V_{NR}$, we have $r^* < r_{NR}$.

From (11c) and $s_t = B_t/(1 + B_t)$, we have $\psi_t - r_t = \mu(1 + B^*)$ (> 0). Because the growth rate is given by (13), we have:

$$g_t = \frac{1 - V^*}{\sigma^2}(1 + B^*) - \delta(\zeta^*) \equiv g^*. \tag{F.8}$$

From (E.5) and $0 < V^* < V_{NR}$, we have $g^* < g_{NR}$.

G Proof of Proposition 4

Proof of (i): From (4) and (15), we have:

$$\frac{A^2}{\delta_1} = \left(\frac{\rho}{V^2} + \frac{1-V}{\sigma^2 V} \right) (1+B), \tag{G.1}$$

where $(V, B) = (V^*, B^*)$ or (V_L, B_L). Since $B^* > 0$, the above equation implies:

$$\frac{\rho}{V^{*2}} + \frac{1-V^*}{\sigma^2 V^*} < \frac{\rho}{{V_L}^2} + \frac{1-V_L}{\sigma^2 V_L}.$$

The LHS decreases with V^*. Thus, we have $V^* > V_L$. From (E.2) and (F.6), we have $q^* > q_L$. From (E.3) and (F.7), we have $r^* > r_L$. From (E.1) and (F.2), we have $\zeta^* > \zeta_L$.

Proof of (ii): Since $Y_t = A\zeta_t K_t$ and $\zeta^* > \zeta_L$, we have $Y_t^* > Y_{L,t}$ if both steady states have the same level of K_t.

We rearrange (G.1) as:

$$\frac{1+B}{V} = \frac{A^2/\delta_1}{\frac{\rho}{V} + \frac{1-V}{\sigma^2}}, \tag{G.2}$$

where $(V, B) = (V^*, B^*)$ or $(V_L, 0)$. Thus, we have:

$$C_t = \rho\omega_t = \rho \frac{1+B}{V} K_t = \rho \frac{A^2/\delta_1}{\frac{\rho}{V} + \frac{1-V}{\sigma^2}} K_t. \tag{G.3}$$

The last term increases with V. Thus, we have $C_t^* > C_{L,t}$ if both steady states have the same level of capital stock.

(20c) is proven as follows. From $Y_t = C_t + I_t$, $Y_t = A\zeta_t K_t$, (4), and (G.3), we have:

$$I_t = \frac{A^2}{\delta_1} V \frac{(1-V)V}{\rho\sigma^2 + (1-V)V} K_t. \tag{G.4}$$

Note that V_L is a positive solution of (17) that can be rewritten as:

$$\Lambda(V) \equiv (\sigma^2 A^2 + \delta_1)V^2 - \delta_1 V - \rho\delta_1\sigma^2 = 0.$$

Thus, we have:

$$V_L = \frac{\delta_1 + \sqrt{{\delta_1}^2 + 4\rho\delta_1\sigma^2(\sigma^2 A^2 + \delta_1)}}{2(\sigma^2 A^2 + \delta_1)} \to \frac{\delta_1}{\sigma^2 A^2 + \delta_1} \left(\equiv V_{L,\rho\to 0}\right), \tag{G.5}$$

as $\rho \to 0$. Further, we have:

$$V^* = 1 - \sigma\sqrt{\rho + \mu} \to 1 - \sigma\sqrt{\mu} \left(\equiv V^*_{\rho\to 0}\right), \quad \text{as} \quad \rho \to 0. \tag{G.6}$$

Thus, both V_L and V^* converge to a constant as $\rho \to 0$. From (G.4), we have:

$$I_t \to \frac{A^2}{\delta_1} V_{\rho\to 0} K_t, \quad \text{where} \quad V_{\rho\to 0} = V^*_{\rho\to 0} \quad \text{or} \quad V_{L,\rho\to 0}.$$

Since $V^* > V_L$, (20c) holds if $\rho > 0$ is sufficiently small.

Proof of (iii): If we use (2), (4), and (G.4), the growth rate is written as:

$$g = \frac{I_t}{K_t} - \frac{(AV)^2}{2\delta_1} - \delta_2 = \frac{A^2}{\delta_1}\left(X(V)V - \frac{V^2}{2}\right) - \delta_2, \text{ where } X(V) \equiv \frac{(1-V)V}{\rho\sigma^2 + (1-V)V}. \tag{G.7}$$

Above, we have $(V, g) = (V_L, g_L)$ or (V^*, g^*). We differential the above equation with respect to V:

$$\frac{\partial g}{\partial V} = \frac{A^2}{\delta_1}\left(X(V) + X'(V)V - V\right), \text{ where } X'(V) = \frac{\rho\sigma^2(1-2V)}{\rho\sigma^2 + (1-V)V}. \tag{G.8}$$

Since both V_L and V^* converge to a constant as $\rho \to 0$, we have:

$$\frac{\partial g}{\partial V} \to \frac{A^2}{\delta_1}\left(1 - V_{\rho\to 0}\right) > 0 \text{ for } V_{\rho\to 0} < 1. \tag{G.9}$$

where $V_{\rho\to 0} = V_{L,\rho\to 0}$ or $V^*_{\rho\to 0}$. Thus, since $0 < V_L < V^* < 1$, we have $g^* > g_L$ for sufficiently small $\rho > 0$.

Using (G.7), we can show that $g^* + \delta_2 > 0$ and $g_L + \delta_2 > 0$ as follows:

$$\begin{aligned} g + \delta_2 &= \frac{A^2}{\delta_1}V\left(X(V) - \frac{V}{2}\right) \\ &> \frac{A^2}{\delta_1}V\left(X(V) - V\right) = \frac{A^2}{\delta_1}V^2\frac{(1-V)^2 - \sigma^2\rho}{\rho\sigma^2 + (1-V)V} \\ &> 0, \end{aligned} \tag{G.10}$$

where $(V, g) = (V_L, g_L)$ or (V^*, g^*). The last inequality holds because $0 < V_L < V^* = 1 - \sigma(\rho+\mu)^{1/2} < 1$. (G.10) shows that if $\delta_2 \geq 0$ is sufficiently small, we have $g_L > 0$.

H Proof of Proposition 5

We first derive $W_L(k_{i,t})$. We have $\omega_t = K_t/V_t + p_t M = (1 + B_t)K_t/V_t$. Since both V_t and B_t are constant in the steady state, we have:

$$g_t = \frac{\dot{K}_t}{K_t} = \frac{\dot{\omega}_t}{\omega_t} = r_t(1 - s_t) + \psi_t s_t + \left(\frac{1 - V_t}{\sigma}\right)^2 - \rho. \tag{H.1}$$

To obtain the last equality, we aggregate (11d) over i, using the facts that $\omega_{i,t}$ and $dW_{i,t}$ are independent, and that $dW_{i,t}$ follows a normal distribution with zero mean.

Since $g_L = (1 - V_L)/\sigma^2 - \delta(\zeta_L)$ holds in the bubbleless economy, we have:

$$\frac{1 - V_L}{\sigma} = \sigma\left(g_L + \delta(\zeta_L)\right). \tag{H.2}$$

From $s_t = 0$, (B.5), (B.6), (H.2) and $U(a_{i,t}, t) = D(\log a_{i,t} + u_t)$, we have:

$$\rho U(a_{i,t}, t) = \log a_{i,t} + \log\rho + \frac{1}{\rho}\left[r_L + \left(\frac{1 - V_L}{\sigma}\right)^2 - \rho\right] - \frac{1}{2\rho}\left\{\sigma\left(g_L + \delta(\zeta_L)\right)\right\}^2. \tag{H.3}$$

At the steady state, we have $\dot{u}_t = 0$. Thus, from (H.1) and (H.3), we obtain:

$$\rho U(a_{i,t}, t) = \log \rho a_{i,t} + \frac{1}{\rho}\left[g_L - \frac{1}{2}\left\{\sigma\left(g_L + \delta(\zeta_L)\right)\right\}^2\right]. \tag{H.4}$$

Since $\rho a_{i,t} = \rho v_L k_{i,t} = (C_{L,t}/K_t)k_{i,t}$ in the bubbleless steady state, (H.4) is rewritten as:

$$\rho W_L(k_{i,t}) = \log \frac{C_{L,t}}{K_t} k_{i,t} + \frac{g_L}{\rho} - \frac{1}{2\rho}\sigma^2(g_L + \delta(\zeta_L))^2. \tag{H.5}$$

The term $-\frac{\sigma^2}{2\rho}(g_L + \delta(\zeta_L))^2$ captures the utility loss from investment risks.

Next, we derive $W^*(k_{i,t})$. Since $g^* = (1 - V^*)(1 + B^*)/\sigma^2 - \delta(\zeta^*)$ holds in the bubbly steady state, we have:

$$\frac{1 - V^*}{\sigma} = \frac{\sigma\left(g^* + \delta(\zeta^*)\right)}{1 + B^*}. \tag{H.6}$$

At the steady state, we have $\dot{u}^*_t = 0$. Thus, from (B.13), (B.14), (H.6) and $U^*(\omega_{i,t}, t) = D^*(\log \omega_{i,t} + u^*_t)$, we have:

$$\begin{aligned}\rho U^*(\omega_{i,t}, t) = \log \omega_{i,t} + \log \rho + \frac{1}{\rho}\left[r^*(1 - s^*) + \psi s^* + \left(\frac{1 - V^*}{\sigma}\right)^2 - \rho\right] \\ - \frac{1}{2\rho}\left\{\frac{\sigma\left(g^* + \delta(\zeta^*)\right)}{1 + B^*}\right\}^2 - \mu\left[U^*(\omega_{i,t}, t) - U(a_{i,t}, t)\right].\end{aligned} \tag{H.7}$$

From (H.1) and (H.7), we obtain:

$$\rho U^*(\omega_{i,t}, t) = \log \rho\omega_{i,t} + \frac{1}{\rho}\left[g^* - \frac{1}{2}\left\{\frac{\sigma\left(g^* + \delta(\zeta^*)\right)}{1 + B^*}\right\}^2\right] - \mu\left[U^*(\omega_{i,t}, t) - U(a_{i,t}, t)\right]. \tag{H.8}$$

In the bubbly steady state, we have:

$$\rho\omega_{i,t} = \rho\omega_t \frac{\omega_{i,t}}{\omega_t} = C^*_t \frac{v^* k_{i,t}}{v^* K_t} = \frac{C^*_t}{K_t} k_{i,t}. \tag{H.9}$$

The second equality uses $a_{i,t} = v^* k_{i,t} = (1 - s^*)\omega_{i,t}$, $K_t = \int_0^1 k_{i,t} di$ and $\omega_t = \int_0^1 \omega_{i,t} di$. From (H.8) and (H.9), we obtain:

$$\rho W^*(k_{i,t}) = \log \frac{C^*_t}{K_t} k_{i,t} + \frac{g^*}{\rho} - \frac{1}{2\rho}\left\{\frac{\sigma\left(g^* + \delta(\zeta^*)\right)}{1 + B^*}\right\}^2 - \mu\left[W^*(k_{i,t}) - W_L(k_{i,t})\right]. \tag{H.10}$$

The last term represents the utility loss owing to a bubble crash. The term $-\frac{\sigma^2}{2\rho}\left(\frac{g^* + \delta(\zeta^*)}{1 + B^*}\right)^2$ captures the utility loss from investment risks.

We now prove Proposition 5. From (H.5) and (H.10), we obtain:

$$(\rho + \mu)\left[W^*(k_{i,t}) - W_L(k_{i,t})\right] = \log \frac{C^*_t}{C_{L,t}} + \frac{g^* - g_L}{\rho} + \frac{\sigma^2}{2\rho}\left[(g_L + \delta(\zeta_L))^2 - \left(\frac{g^* + \delta(\zeta^*)}{1 + B^*}\right)^2\right]. \tag{H.11}$$

From Proposition 4, the first and second terms in (H.11) are positive. From (H.2) and (H.6), we have:

$$\text{sign}\ \left\{(g_L+\delta(\zeta_L))^2-\left(\frac{g^*+\delta(\zeta^*)}{1+B^*}\right)^2\right\} = \text{sign}\ \left\{(1-V_L)^2-(1-V^*)^2\right\}.$$

Since $0<V_L<V^*<1$ holds, we have $(1-V_L)^2>(1-V^*)^2$.

I Proof of Proposition 6

Proof of (i): If $\hat{\varepsilon}>\delta_2$, we have $r_t\geq-\delta_2>-\hat{\varepsilon}$. Thus, underutilization never occurs.

Proof of (ii): Consider a bubbleless economy ($B_t=0$). We denote the general good price in the bubbleless steady state with underutilization as V_U.

We first show that $V_U>V_L$, where V_L is defined in Proposition 2. In an equilibrium with underutilization, the utilization rate is determined by (23). In this case, the condition $\zeta<AV_U/\delta_1$ must hold. Thus, V_U satisfies:

$$\frac{A^2}{\delta_1}V_U>\frac{\rho}{V_U}+\frac{1-V_U}{\sigma^2}.$$

Since V_L is a unique positive solution of $A^2V/\delta_1=\rho/V+(1-V)/\sigma^2$, the above equation indicates that:

$$V_U>V_L. \tag{I.1}$$

We substitute (23) into (24) to eliminate ζ. Then, we have:

$$\frac{(1-V)V}{\sigma^2}=\frac{\delta_1}{2A^2}\left(\frac{\rho}{V}+\frac{1-V}{\sigma^2}\right)^2+\delta_2-\hat{\varepsilon}-\rho. \tag{I.2}$$

The LHS is a concave function of V, as shown in Figure A1, while the RHS is a decreasing and convex function of V. The equation determines V_U. If both sides of (I.2) intersect at just one point, this intersection gives V_U (see Figure A1 (a)). If the equation has two intersections, the left intersection is the equilibrium, which determines V_U, while the right one (point A) is not an equilibrium (see Figure A1 (b)). Suppose point A is an equilibrium. Since point A is a solution to (I.2), underutilization occurs if the general good firm chooses ζ_t according to (4). In this case, downward pressure is exerted on V. If V decreases slightly from V_A, the LHS of (I.2) becomes larger than the RHS, meaning that $r>-\hat{\varepsilon}$ holds and the ZLB is not binding. Thus, V can fall below V_A, and point A is not an equilibrium.

Without intersections, as shown in Figure A1 (c), the LHS < the RHS for all $V\in(0,1)$. This condition implies $r<\hat{\varepsilon}$, meaning $r\geq-\hat{\varepsilon}$ is never satisfied. Thus, no equilibria exist. As $\hat{\varepsilon}$ increases, the RHS shifts downward, leading to the conclusion that for sufficiently small $\hat{\varepsilon}$, no equilibria exist.

Next, we derive the conditions under which both sides of (I.2) have a unique intersection. We begin by deriving the conditions under which $\text{LHS}|_{V=V_L}<\text{RHS}|_{V=V_L}$ holds. Since V_L is the unique positive solution to the equation $A^2V/\delta_1=\rho/V+(1-V)/\sigma^2$, we have:

$$\text{LHS}|_{V=V_L}=\frac{A^2V_L^2}{\delta_1}-\rho\ \text{ and }\ \text{RHS}|_{V=V_L}=\frac{A^2V_L^2}{2\delta_1}+\delta_2-\hat{\varepsilon}-\rho.$$

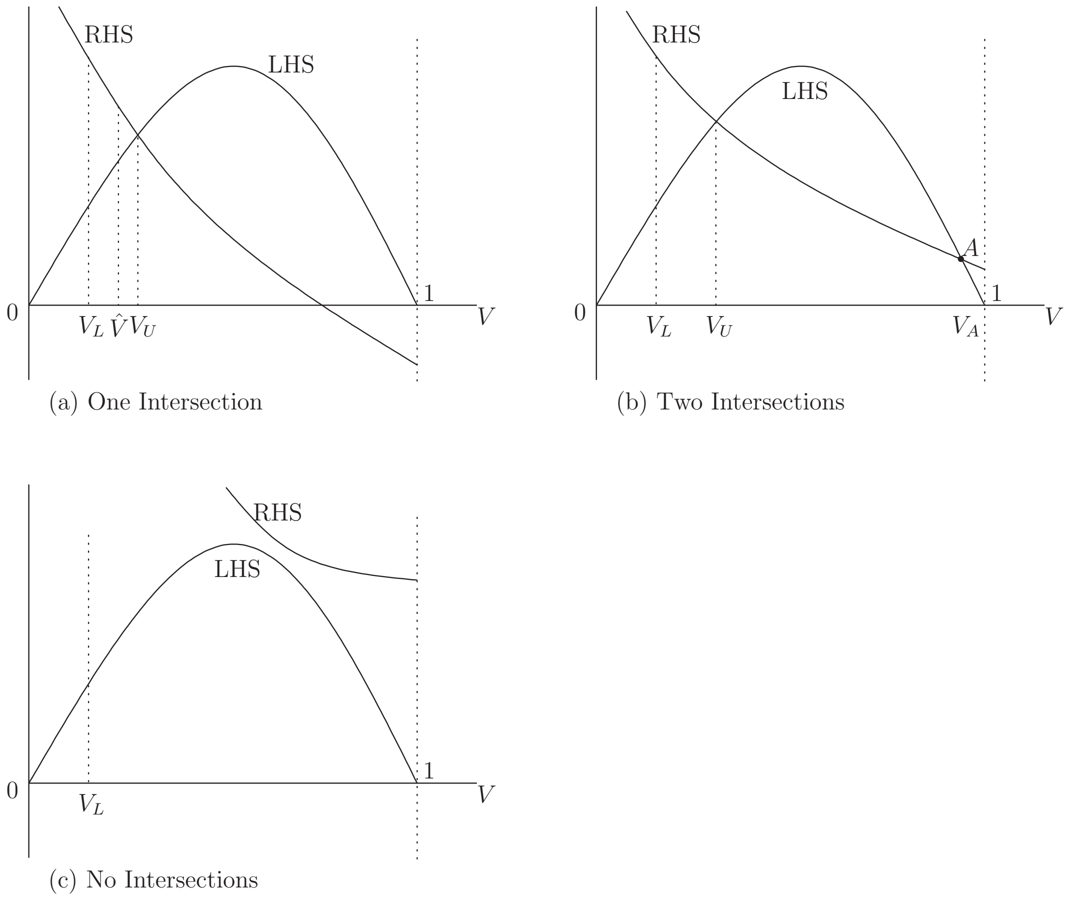


Figure A1 Existence of the Bubbleless Steady State with the ZLB

Thus, inequality $\text{LHS}|_{V=V_L} < \text{RHS}|_{V=V_L}$ is equivalent to $(AV_L)^2/(2\delta_1) - \delta_2 + \hat{\varepsilon} = r_L + \hat{\varepsilon} < 0$. This inequality holds if and only if $V_L < \hat{V}$ (see (22)), or equivalently:

$$\frac{A^2\hat{V}}{\delta_1} > \frac{\rho}{\hat{V}} + \frac{1-\hat{V}}{\sigma^2}. \tag{I.3}$$

(Recall that V_L is a unique positive solution of $A^2V/\delta_1 = \rho/V + (1-V)/\sigma^2$.) We substitute the definition of $\hat{V}$ in (22) into the above equation. After some manipulation, we obtain:

$$[2(\delta_2 - \hat{\varepsilon}) - \rho]\sigma^2 \cdot A^2 - \sqrt{2\delta_1(\delta_2 - \hat{\varepsilon})} \cdot A + 2\delta_1(\delta_2 - \hat{\varepsilon}) > 0.$$

Given that $2(\delta_2 - \hat{\varepsilon}) - \rho > 0$ (or $\hat{\varepsilon} < \delta_2 - \rho/2$), the above inequality holds if A is sufficiently large.

We next evaluate both sides of (I.2) at $V = 1$:

$$\text{LHS}|_{V=1} = 0 \quad \text{and} \quad \text{RHS}|_{V=1} = \frac{\delta_1\rho^2}{2A^2} + \delta_2 - \hat{\varepsilon} - \rho.$$

Thus, we have $\text{LHS}|_{V=1} = 0 > \text{RHS}|_{V=1}$ if and only if:

$$\delta_2 - \rho + \frac{\delta_1\rho^2}{2A^2} < \hat{\varepsilon}.$$

From (18), we know that the inequality $\delta_2 - \rho + \frac{\delta_1\rho^2}{2A^2} < \delta_2 - \rho/2$ holds. Thus, we can conclude that if $\delta_2 - \rho + \frac{\delta_1\rho^2}{2A^2} < \hat{\varepsilon} < \delta_2 - \rho/2$ and A is sufficiently large, both sides of (I.2) will have a unique intersection, as shown in Figure A1 (a). In this case, a unique bubbleless steady state with underutilization exists.

If $\hat{\varepsilon}$ decreases slightly below $\delta_2-\rho+\frac{\delta_1\rho^2}{2A^2}$, two intersections occur, with the left intersection providing a unique equilibrium, as shown in Figure A1 (b). If $\hat{\varepsilon}$ decreases further, the intersection disappears, resulting in no equilibrium, as shown in Figure A1 (c). Thus, a lower bound $\underline{\varepsilon}(< \delta_2 - \rho + \frac{\delta_1\rho^2}{2A^2})$ exists. If $\underline{\varepsilon} < \hat{\varepsilon} < \delta_2 - \rho/2$ and A is sufficiently large, a unique bubbleless steady state with underutilization emerges.

Proof of (iii): On one hand, $V^* = 1 - \sigma\sqrt{\rho+\mu}$ is independent of A. On the other hand, $\hat{V}$, as defined in (22), decreases with A. We have the limits $\lim_{A\to 0}\hat{V} = +\infty$ and $\lim_{A\to\infty}\hat{V} = 0$. Thus, for sufficiently large A, we have $V^* > \hat{V}$, implying that the ZLB becomes irrelevant.

J Proof of Proposition 7

When $B = 0$, both C_t/K_t and I_t/K_t are decreasing functions of V. Since $V_U > V_L$ holds (see (I.1)), we have $C_U/K_t < C_L/K_t$ and $I_U/K_t < I_L/K_t$. From $Y_t = C_t + I_t$, we have $Y_U/K_t < Y_L/K_t$. Because $Y_t = A\zeta_t K_t$, we have $\zeta_U < \zeta_L$.

The growth rate of the bubbleless economy is given by $g = (1-V)/\sigma^2 - \delta(\zeta)$. Irrespective of the presence of the ZLB, V and ζ satisfy $A\zeta = \frac{\rho}{V} + \frac{1-V}{\sigma^2}$, which implies:

$$\frac{1-V}{\sigma^2} = \frac{A\zeta}{\frac{\rho}{V} + \frac{1-V}{\sigma^2}}\frac{1-V}{\sigma^2} = \frac{A\zeta V(1-V)}{\rho\sigma^2 + V(1-V)},$$

where $(\zeta, V) = (\zeta_L, V_L)$ or (ζ_U, V_U). Thus, we have:

$$g = A\zeta\frac{V(1-V)}{\rho\sigma^2 + V(1-V)} - \delta(\zeta). \tag{J.1}$$

We know that as $\rho \to 0$, V_L converges to $V_{L,\rho\to 0}$ (see Appendix G). From (4), ζ_L converges to $\zeta_{L,\rho\to 0} \equiv AV_{L,\rho\to 0}/\delta_1$. Since V_U is the smaller solution of (I.2), V_U also converges to a constant, which we denote as $V_{U,\rho\to 0}$. Further, ζ_U converges to $\zeta_{U,\rho\to 0} \equiv \frac{1-V_{U,\rho\to 0}}{\sigma^2 A}$. Thus, we have $g \to A\zeta_{\rho\to 0} - \delta(\zeta_{\rho\to 0})$, as $\rho \to 0$, where $\zeta_{\rho\to 0} = \zeta_{L,\rho\to 0}$ or $\zeta_{U,\rho\to 0}$. We have:

$$\frac{\partial}{\partial\zeta_{\rho\to 0}}\left(A\zeta_{\rho\to 0} - \delta(\zeta_{\rho\to 0})\right) = A - \delta_1\zeta_{\rho\to 0} > 0, \quad \text{if} \quad \zeta_{\rho\to 0} < A/\delta_1.$$

Since $\zeta_U < \zeta_L \equiv AV_L/\delta_1 < A/\delta_1$, we have $g_U < g_L$ if $\rho > 0$ is sufficiently small.

K Proof of Proposition 8

If we substitute $B_t = 0$, (29) and $V_t = V_L^\tau$ into (27), we obtain:

$$\frac{A^2 V_L^\tau}{\delta_1} = \left[\frac{\rho}{V_L^\tau} + \frac{1-V_L^\tau}{\sigma^2}\right](1+B^*), \tag{K.1}$$

where B^* is defined in Proposition 3, this equation determines V_L^τ. The comparison between (K.1) and (15) reveals that $V_L^\tau = V^*$. From $V_L^\tau = V^*$, (5), (4), and (8), we immediately obtain that $\zeta_L^\tau = \zeta^*$, $q_L^\tau = q^*$, and $r_L^\tau = r^*$, respectively. Since underutilization does not occur in the bubbly steady state, it also does not occur in the bubbleless steady state with policies. From $Y_t = A\zeta_t K_t$ and $\zeta_L^\tau = \zeta^*$, we have $Y_{L,t}^\tau / K_t = Y_t^* / K_t$. After aggregating (26a) and (26b), we use $V^* = V_L^\tau$, $\omega_t = \upsilon_t K_t$, and (29) to obtain:

$$\frac{C_{L,t}^\tau}{K_t} = \frac{\rho}{V^*}(1 + B^*) = \frac{C_t^*}{K_t} \quad \textit{and} \quad \frac{I_{L,t}^\tau}{K_t} = \frac{1 - V^*}{\sigma^2}(1 + B^*) = \frac{I_t^*}{K_t}.$$

From (13), $I_{L,t}^\tau / K_t = I_t^* / K_t$, and $\zeta_L^\tau = \zeta^*$, we have $g_L^\tau = g^*$.

L Price Elasticity of the Aggregate Supply Curve : $\eta \neq 1$

This appendix demonstrates that even if $\eta \neq 1$, our main results remain unaffected. With $\eta \neq 1$, the utilization rate and the general good supply are given by (4) and the first equality of (6), respectively. These equations are reproduced here:

$$\zeta_t = \left(\frac{AV_t}{\delta_1}\right)^\eta \equiv \zeta(V_t), \tag{L.1}$$

$$Y_t = A^{1+\eta} \left(\frac{V_t}{\delta_1}\right)^\eta K_t. \tag{L.2}$$

The general good market clearing condition is not affected. The dynamics of B_t follow (16). Let us reproduce these equations again:

$$A\zeta_t = \left(\frac{\rho}{V_t} + \frac{1 - V_t}{\sigma^2}\right)(1 + B_t), \tag{L.3}$$

$$\dot{B}_t = \left[\mu(1 + B_t) + A\zeta_t V_t - \frac{1 - V_t}{\sigma^2}(1 + B_t)\right] B_t, \tag{L.4}$$

Using (L.1), (L.3), and $B_t = 0$, we show the existence of the bubbleless steady state. The following proposition is analogous to Proposition 2:

Proposition A1 *Suppose that $\sigma > 0$. If and only if $\delta_1 > 0$ is sufficiently small to satisfy:*

$$\delta_1{}^\eta \rho < A^{1+\eta}, \tag{L.5}$$

(L.3) *has a unique positive solution $V_L \in (0, 1)$. Then, a unique bubbleless steady-state equilibrium exists, where $I_t > 0$ and $V_t = V_L$, $B_t = 0$, $\zeta_t = \zeta(V_L) \equiv \zeta_L (< \zeta_{NR})$, $q_t = A\zeta_L - \delta(\zeta_L)/V_L = A^{1+\eta}/(1+\eta)(V_L/\delta_1)^\eta - \delta_2/V_L \equiv q_L$, $r_t = q_L V_L = (AV_L)^{1+\eta}/[(1+\eta)\delta_1{}^\eta] - \delta_2 \equiv r_L$, and $g_t = (1 - V_L)/\sigma^2 - \delta(\zeta_L) \equiv g_L$. In the bubbleless economy, the bubbleless steady-state equilibrium is the only equilibrium.*

(Proof) As V increases from 0 to 1, the LHS of (L.3) monotonically increases from 0 to $A^{1+\eta}/\delta_1{}^\eta$, whereas The RHS decreases from $+\infty$ to ρ. Thus, (L.5) ensures that (L.3) has a unique solution $V_L \in (0, 1)$. □

The existence of the bubbly steady state is shown by the following proposition that is analogous to Proposition 3.

Proposition A2 *Suppose that $\sigma > 0$ and that the following two inequalities hold:*

$$0 < \sigma < (\rho + \mu)^{-\frac{1}{2}} \qquad \textit{and} \qquad {\delta_1}^{\eta} < A^{1+\eta} V^{*\eta} \cdot Z, \tag{L.6}$$

where

$$V^* \equiv 1 - \sigma(\rho+\mu)^{\frac{1}{2}} \ \in (0,\ 1) \qquad \textit{and} \qquad Z \equiv \frac{1 - \sigma(\rho+\mu)^{\frac{1}{2}}}{\frac{1}{\sigma}(\rho+\mu)^{\frac{1}{2}} - \mu}. \tag{L.7}$$

Then, a unique bubbly steady-state equilibrium exists, where $I_t > 0$ and $V_t = V^* \in (0,1)$, $\zeta_t = \zeta(V^*) \equiv \zeta^*$, $B_t = ZA\zeta^* - 1 \equiv B^*(> 0)$, $q_t = A\zeta^* - \delta(\zeta^*)/V^* = A^{1+\eta}/(1+\eta)(V^*/\delta_1)^{\eta} - \delta_2/V^* \equiv q^*$, $r_t = q^*V^* = (AV^*)^{1+\eta}/[(1+\eta){\delta_1}^{\eta}] - \delta_2 \equiv r^*$, $\psi_t - r_t = \mu(1+B^*)(> 0)$, and $g_t = \frac{1-V^*}{\sigma^2}(1+B^*) - \delta(\zeta^*) \equiv g^*$.

(Proof) Set $\dot{B}_t = 0$ in (L.4). Then, solving (L.3) and (L.4) yield $V^* = 1 - \sigma(\rho+\mu)^{\frac{1}{2}}$ and $B^* = ZA\zeta^* - 1$. □

The comparison between the bubbly and bubbleless steady states leads to the following proposition:

Proposition A3 *Suppose that both the bubbly and the bubbleless steady-state equilibria exist.*
(i) Capital utilization: We have $V^* > V_L$, $q^* > q_L$, $r^* > r_L$, and $\zeta^* > \zeta_L$.
(ii) Level effects: Suppose that both steady states have the same level of capital stock at time t, $K_t^* = K_{L,t}$. Then, we have $Y_t^* > Y_{L,t}$. If $V_L > 1/2$, we have $C_t^* > C_{L,t}$. If $\rho > 0$ is sufficiently small, we have $I_t^* > I_{L,t}$.
(iii) Growth effects: If $\rho > 0$ is sufficiently small, we have $g^* > g_L$.

(Proof) Both V^* and V_L satisfy (L.3). Since $B^* > 0$, we have $V^* > V^L$. Notice that $q = A^{1+\eta}/(1+\eta)(V/\delta_1)^{\eta} - \delta_2/V$ and $r = (AV)^{1+\eta}/[(1+\eta){\delta_1}^{\eta}] - \delta_2$ are increasing functions of V. Also, ζ and Y increase with V (see (L.1) and (L.2)). Thus, we have $q^* > q_L$, $r^* > r_L$, $\zeta^* > \zeta_L$, and $Y_t^* > Y_{L,t}$.

If we follow the derivation of (G.3), we have:

$$C_t = \rho \frac{\frac{A^{1+\eta}}{{\delta_1}^{\eta}} V^{\eta-1}}{\frac{\rho}{V} + \frac{1-V}{\sigma^2}} K_t.$$

After taking a logarithm in the above equation, we differentiate it with respect to V and then we obtain:

$$\frac{d \ln C_t}{dV} = \eta \left(\frac{\rho}{V} + \frac{1-V}{\sigma^2} \right) + \frac{2V-1}{\sigma^2}.$$

Then, $\frac{d \ln C_t}{dV} > 0$ holds for $V > 1/2$. Thus, if $V_L > 1/2$, we have $C^* > C_L$.

Similarly to (G.4), we have:

$$I_t = \frac{A^{1+\eta}}{{\delta_1}^{\eta}} V^{\eta} \frac{(1-V)V}{\rho\sigma^2 + (1-V)V} K_t. \tag{L.8}$$

Note that V_L is a positive solution of:

$$A^{1+\eta}\left(\frac{V}{\delta_1}\right)^{\eta} = \frac{\rho}{V_t} + \frac{1-V_t}{\sigma^2}.$$

Thus, as $\rho \to 0$, V_L converges to a constant $V_{L,\rho\to 0}$. Further, we have:

$$V^* = 1 - \sigma\sqrt{\rho+\mu} \to 1 - \sigma\sqrt{\mu}\left(\equiv V^*_{\rho\to 0}\right), \quad \text{as} \quad \rho \to 0.$$

Thus, both V_L and V^* converge to a constant as $\rho \to 0$. From (L.8), we have:

$$I_t \to \frac{A^{1+\eta}}{{\delta_1}^{\eta}} {V_{\rho\to 0}}^{\eta} K_t, \quad \text{where} \quad V_{\rho\to 0} = V^*_{\rho\to 0} \quad \text{or} \quad V_{L,\rho\to 0}.$$

Since $V^* > V_L$, we have $I^*_t > I_{L,t}$ if $\rho > 0$ is sufficiently small.

Like (G.7), the growth rate is written as:

$$g = \frac{A^{1+\eta}}{{\delta_1}^{\eta}}\left(X(V)V^{\eta} - \frac{V^{1+\eta}}{1+1/\eta}\right) - \delta_2, \text{ where } X(V) \equiv \frac{(1-V)V}{\rho\sigma^2 + (1-V)V},$$

where $(V, g) = (V_L, g_L)$ or (V^*, g^*). We differential the above equation with respect to V:

$$\frac{\partial g}{\partial V} = \frac{A^{1+\eta}}{{\delta_1}^{\eta}}\left(\eta X(V)V^{\eta-1} + X'(V)V^{\eta} - \eta V^{\eta}\right), \text{ where } X'(V) = \frac{\rho\sigma^2(1-2V)}{\rho\sigma^2 + (1-V)V}.$$

Since both V_L and V^* converge to a constant as $\rho \to 0$, we have:

$$\frac{\partial g}{\partial V} \to \frac{A^{1+\eta}}{{\delta_1}^{\eta}}\left(1 - V_{\rho\to 0}\right)\eta V^{\eta-1} > 0 \text{ for } V_{\rho\to 0} < 1.$$

where $V_{\rho\to 0} = V_{L,\rho\to 0}$ or $V^*_{\rho\to 0}$. Thus, since $0 < V_L < V^* < 1$, we have $g^* > g_L$ for sufficiently small $\rho > 0$. □

Dynamics after Bubble Crash: Proposition 4 holds even if $\eta \neq 1$, except when $C^*_t > C_{L,t}$. To prove that $C^*_t > C_{L,t}$, we need an additional condition: $V_L > 1/2$. When $V_L > 1/2$ is met, a collapse of the bubble leads to a demand-driven recession, as discussed in Section 5.

ZLB: Proposition A3 shows that a collapse of bubbles puts downward pressure on the general good price V and the interest rate r. As discussed in Section 6, this bubble crash leads to a binding ZLB, causing underutilization of capital.

Aggregate Demand Policy: With the consumption subsidy τ_c and the public insurance τ_I, (L.3) is modified as:

$$A\zeta_t = \left[\frac{1}{(1-\tau_c)}\frac{\rho}{V_t} + (1+\tau_I)\frac{1-V_t}{\sigma^2}\right](1+B_t),$$

which is exactly the same as (27). Clearly, an increase in τ_c, stimulating the aggregate demand, increases the general good production. Furthermore, if τ_c and τ_σ satisfy:

$$\frac{1}{1-\tau_c} = 1 + \tau_I = 1 + B^*,$$

then, the bubbleless economy achieves the same allocation as the bubbly economy.

M Neoclassical Technology

This appendix demonstrates that the main results from the AK model hold when using a neoclassical production function. Entrepreneurs solve the same utility maximization problem as in the AK model, so (11a)-(11d) and (12a)-(12c) remain valid.

Production sector: Consider the following production technology:

$$Y_t = F(\zeta_t K_t, A_t L_t), \qquad A_t > 0, \tag{M.1}$$

where A_t grows over time at an exogenous constant rate $g \equiv \dot{A}_t/A_t > 0$ and L_t represents labor demand. We assume that F is continuous, exhibits constant returns to scale, and has positive and diminishing marginal products in both K and L, satisfying the Inada conditions. Additionally, we assume $F(0, A_t L_t) = F(\zeta_t K_t, 0) = 0$. In (M.1), the utilization rate ζ_t works like capital-augmenting productivity. Thus, we assume that the marginal product of capital, $\partial F(\zeta_t K_t, A_t L_t)/\partial K_t = \zeta_t \partial F(\zeta_t K_t, A_t L_t)/\partial(\zeta_t K_t)$, increases with ζ_t.

The profit of the firm is given by $\Pi_t^Y = F(\zeta_t K_t, A_t L_t) - w_t L_t - q_t K_t - \delta(\zeta_t) v_t K_t$, where w_t denote the wage rate and $\delta(\zeta_t)$ is given by (2). We define $k_t \equiv K_t/(A_t L_t)$ and $y_t \equiv Y_t/(A_t L_t) = F(\zeta_t K_t/(A_t L_t), 1) = f(\zeta_t k_t)$. All markets are competitive and the profit maximization gives the following equations:

$$q_t = \zeta_t f'(\zeta_t k_t) - \delta(\zeta_t) v_t, \tag{M.2}$$

$$w_t = A_t \left[f(\zeta_t k_t) - f'(\zeta_t k_t)\zeta_t k_t \right], \tag{M.3}$$

$$f'(\zeta_t k_t) = \delta'(\zeta_t) v_t, \tag{M.4}$$

where $f'(\zeta_t k_t) \equiv \frac{df(\zeta_t k_t)}{d\zeta_t k_t}$ and $\delta'(\zeta_t) \equiv \frac{d\delta(\zeta_t)}{d\zeta_t}$.

Workers: The number of workers is denoted as L, which remains constant over time. Workers supply one unit of labor inelastically and earn a wage rate w_t. In equilibrium, $L_t = L$ holds. We assume that workers consume their entire labor income in a hand-to-mouth manner. The aggregate consumption of workers, C_t^w, is given by:

$$C_t^w = w_t L. \tag{M.5}$$

Equilibrium dynamics: The following proposition characterizes equilibrium dynamics.

Proposition A4 *Suppose that $\sigma > 0$. In an equilibrium where $I_t > 0$ holds, V_t, B_t, k_t, and ζ_t are determined by the following four equations:*

$$f'(\zeta_t k_t)\zeta_t = \left(\frac{\rho}{V_t} + \frac{1 - V_t}{\sigma^2} \right)(1 + B_t), \tag{M.6}$$

$$\dot{B}_t = \left\{ \mu(1 + B_t) + f'(\zeta_t k_t)\zeta_t V_t - \frac{1 - V_t}{\sigma^2}(1 + B_t) \right\}, \tag{M.7}$$

$$\dot{k}_t = \left\{ \frac{1 - V_t}{\sigma^2}(1 + B_t) - \delta(\zeta_t) - g \right\} k_t, \tag{M.8}$$

$$\delta'(\zeta_t) = f'(\zeta_t k_t) V_t. \tag{M.9}$$

Proof: See Appendix M.1.

Existence of the bubbly steady-state equilibrium: As in the AK model, the subscript L and the asterisk $*$ denote the values for the bubbleless and bubbly steady-state equilibria, respectively. The following proposition demonstrates the existence of both the bubbleless and bubbly steady-state equilibria.

Proposition A5 *Suppose that $\sigma > 0$, $(1+\eta)(g+\delta_2) - \eta\mu > 0$, and δ_2 is sufficiently small. Define $\underline{\sigma} \equiv \frac{(\rho+\mu)^{\frac{1}{2}}}{(1+\eta)(g+\delta_2)-\eta\mu}$ and $\overline{\sigma} \equiv \left[g + \delta_2 + \left(\frac{\eta}{1+\eta}\right)\rho\right]^{-\frac{1}{2}}$. Then, the following holds.*
(i) If $\sigma \leq \underline{\sigma}$, then only the bubbleless steady-state equilibrium exists.
(ii) If $\underline{\sigma} < \sigma < \overline{\sigma}$, then both bubbly and bubbleless steady-state equilibria exist.
(iii) If $\overline{\sigma} \leq \sigma < (\rho+\mu)^{-\frac{1}{2}}$, then only the bubbly steady-state equilibrium exists.
(iv) If $(\rho+\mu)^{-\frac{1}{2}} \leq \sigma$, then neither a bubbly nor a bubbleless steady-state equilibrium exists.

Proof: See Appendix M.2.

If σ is in an intermediate range (case (ii)), both the bubbly and bubbleless steady states exist.

Comparison between steady states: We define $c_t \equiv C_t/A_tL$, $c_t^w \equiv C_t^w/A_tL$, and $i_t \equiv I_t/A_tL$, respectively. We prove the following proposition.

Proposition A6 *Suppose that the bubbleless and bubbly steady stats exist (the case of (ii) in Proposition A5).*
(i) We have

$$\zeta^* > \zeta_L, \quad V^* > V_L, \quad \text{and} \quad r^* > r_L.$$

(ii) Suppose that σ is sufficiently close to $\overline{\sigma} = \left[g + \delta_2 + \left(\frac{\eta}{1+\eta}\right)\rho\right]^{-\frac{1}{2}}$. Then, we have

$$k^* > k_L, \quad c^* > c_L, \quad c^{w^*} > c_L^w, \quad i^* > i_L, \quad \text{and} \quad y^* > y_L. \tag{M.10}$$

Proof: See Appendix M.3.

Proposition A6 shows that the presence of a bubble increases capital accumulation. Although Propositions 4 and A6 share similarities, they differ in key aspects. Unlike the AK economy, an economy with a neoclassical production function typically has transitional dynamics. After a bubble crash, the economy gradually converges to the new steady state. Thus, the comparison between the two steady states (Proposition A6) does not provide any insight into the immediate impact of the bubble crash.

Dynamics after Bubble Crashes in the Neoclassical Economy: To assess the immediate effects of bubble collapses, we must explore the transitional dynamics. We demonstrate that, similar to the AK model, the collapse of bubbles leads to a demand-driven recession.

Figure A2 shows the phase diagram and illustrates the dynamics of k_t after the collapse of bubbles (see Appendix M.4). Initially, the economy is in the bubbly steady state (point a)

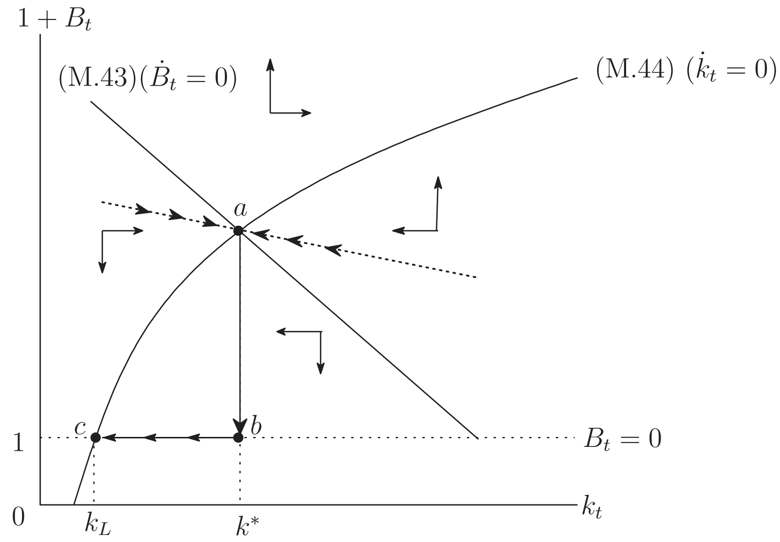


Figure A2 Phase Diagram

in Figure A2. When bubbles collapse ($B^* = 0$), the economy shifts to point b. From there, capital k_t decreases monotonically over time and eventually converges to k_L (point c). The inequality $k_L < k^*$ indicates that the bubble crash leads to a long-term recession.

Appendix M.4 shows that the relative price of the general good, V_t, and the utilization rate, ζ_t, are both increasing functions of B_t. As a result, a collapse of bubbles causes a sudden contraction in both ζ_t and V_t. The underlying reason is straightforward: a bubble crash triggers a negative wealth effect, which immediately reduces aggregate demand.

Since general good production is given by (M.1) and K_t remains constant in the short run, the reduction in utilization leads to a sharp drop in general good production. Following the bubble bust, as capital stock gradually decreases, capital production I_t also contracts. Using (2) and (M.4), we can rewrite the capital rental rate (M.2) as:

$$q_t = \frac{1}{1 + 1/\eta}\zeta_t f'(\zeta_t k_t) - \frac{\delta_2}{V_t}.$$

The capital rental rate increases with V_t. Since the utilization rate, ζ_t, raises the marginal product of capital, $\zeta_t f'(\zeta_t k_t)$, the capital rental rate also rises with ζ_t. As a result, the rental rate drops suddenly at the moment of a bubble collapse.

The effect on entrepreneurs' consumption, however, is ambiguous. In contrast, the collapse of the bubble clearly reduces the labor wage rate and workers' consumption through its impact on utilization (see (M.3) and (M.5)).

Price Rigidity in the Neoclassical Economy: A collapse of bubbles leads to a drop in the general good price, V_t. In the presence of price rigidity, this can result in underutilization, which exacerbates the demand-driven recession, similar to the effects observed in the benchmark AK model.

Demand Policy in the Neoclassical Economy: As discussed in Section 7, consider the consumption subsidy, $\tau_{c,t}$, and the capital creation subsidy, $\tau_{I,t}$. We assume that the subsidy rates vary over time. Entrepreneurs in this extended model behave in the same manner as in the benchmark model. In the bubbleless economy, the equilibrium dynamics are characterized by:

$$f'(\zeta_t k_t)\zeta_t = \frac{1}{1-\tau_{c,t}}\frac{\rho}{V_t} + (1+\tau_{I,t})\frac{1-V_t}{\sigma^2}, \tag{M.11}$$

$$\dot{k}_t = \left\{(1+\tau_{I,t})\frac{1-V_t}{\sigma^2} - \delta(\zeta_t) - g\right\} k_t, \tag{M.12}$$

$$\delta'(\zeta_t) = f'(\zeta_t k_t)V_t. \tag{M.13}$$

Suppose that $\tau_{c,t}$ and $\tau_{I,t}$ satisfy

$$\frac{1}{1-\tau_{c,t}} = 1+\tau_{I,t} = 1+B_t,$$

where B_t represents the asset bubbles in the bubbly economy, with $\tau_{c,t} = \tau_{I,t} = 0$, and follows the dynamic system given by (M.6), (M.7), (M.8), and (M.9). In the bubbleless economy, the dynamics remain the same as in the bubbly economy, except for B_t.

M.1 Proof of Proposition A4

In the neoclassical economy, the market clearing condition for general goods is given by $Y_t = C_t + C_t^w + I_t$. All markets are competitive, and the equation $Y_t = q_t K_t + \delta_t(\zeta_t)v_t K_t + w_t L$ holds. From this equation, (M.2), and (M.5), we obtain $\zeta_t f'(\zeta_t k_t)K_t = C_t + I_t$. Next, similar to the derivation of (15), we can rewrite $\zeta_t f'(\zeta_t k_t)K_t = C_t + I_t$ as (M.6) by using (12a), (12b), (12c), (14), and $V_t = 1/v_t$. (M.7) is derived in the same way as the derivation of (16), except for the term $q_t = \zeta_t f'(\zeta_t k_t) - \delta(\zeta_t)v_t$. From $k_t = K_t/A_t L$, we obtain:

$$\frac{\dot{k}_t}{k_t} = \frac{I_t}{K_t} - \delta(\zeta_t) - g. \tag{M.14}$$

Substituting (12c), (14), and $V_t = 1/v_t$ into (M.14) yields (M.8). Equation (M.13) is derived by (M.4) and $V_t = 1/v_t$. □

M.2 Proof of Proposition A5

First, we prove the following two lemmas, which provide the existence conditions for the bubbleless and bubbly steady-state equilibria, respectively.

Lemma A1 *Suppose that $\sigma > 0$ and δ_2 is sufficiently small. If and only if*

$$\sigma < \frac{1}{\left(g+\delta_2+\frac{\eta}{1+\eta}\rho\right)^{\frac{1}{2}}} \equiv \overline{\sigma} \tag{M.15}$$

holds, the following equation has a unique positive solution $V_L \in (0,1)$

$$\rho + \left(1+\frac{1}{\eta}\right)(g+\delta_2) = \frac{1-V}{\sigma^2}\left(1+\frac{1}{\eta}-V\right). \tag{M.16}$$

Then, a unique bubbleless steady-state equilibrium exists such that $I_t > 0$ *holds and* $V_t = V_L$*,* ζ_t*,* k_t*,* q_t*, and* r_t *satisfy* $\delta_1 \zeta_L^{1+\frac{1}{\eta}} = \rho + (1 - V_L)V_L/\sigma^2$*,* $f'(\zeta_L k_L) = \delta_1 \zeta_L^{\frac{1}{\eta}}/V_L$*,* $q_L = [\zeta_L \delta'(\zeta_L) - \delta(\zeta_L)]/V_L$ (> 0)*,* $r_L = q_L V_L$*, respectively.*

(Proof) Using $B_t = 0$, $\dot{k}_t = 0$, (2), and (M.13), we rewrite (M.6) and (M.8) as

$$\left(1 + \frac{1}{\eta}\right)(\delta(\zeta_t) - \delta_2) = \rho + \frac{(1 - V_t)V_t}{\sigma^2}, \tag{M.17}$$

$$\frac{1 - V_t}{\sigma^2} = \delta(\zeta_t) + g, \tag{M.18}$$

respectively, where we use $\delta'(\zeta_t)\zeta_t = \delta_1 \zeta_t^{1+\frac{1}{\eta}} = \left(1 + \frac{1}{\eta}\right)(\delta(\zeta_t) - \delta_2)$. From (M.17) and (M.18), we have (M.16). The RHS of (M.16) monotonically decreases from $\frac{1}{\sigma^2}\left(1 + \frac{1}{\eta}\right)$ to 0 as V increases from 0 to 1. Thus, (M.16) has a unique positive solution $V_L \in (0, 1)$ if and only if $\rho + \left(1 + \frac{1}{\eta}\right)(g + \delta_2) < \frac{1}{\sigma^2}\left(1 + \frac{1}{\eta}\right)$, or equivalently (M.15). From $V = 1/v$, $B_t = 0$, and (12c), we have $I_t = (1 - V_L)K_t/\sigma^2$. Thus $I_t > 0$ holds if and only if $V_L \in (0, 1)$.

From $B_t = 0$, (2), (M.6), and (M.13), we have $\delta_1 \zeta_L^{1+\frac{1}{\eta}} = \rho + (1 - V_L)V_L/\sigma^2$, which determines ζ_L. From (2) and (M.13), we have $f'(\zeta_L k_L) = \delta_1 \zeta_L^{\frac{1}{\eta}}/V_L$, which gives k_L. The Inada condition ensures that $f'(\zeta_L k_L) = \delta_1 \zeta_L^{\frac{1}{\eta}}/V_L$ has a unique value of k_L. Using $f'(\zeta_L k_L) = \delta_1 \zeta_L^{\frac{1}{\eta}}/V_L$, and $V_L = 1/v_L$, we rewrite (M.2) as

$$q_L = \frac{\zeta_L \delta'(\zeta_L) - \delta(\zeta_L)}{V_L} = \frac{1}{V_L}\left(\frac{1}{1 + \eta}\delta_1 \zeta_L^{1+\frac{1}{\eta}} - \delta_2\right). \tag{M.19}$$

This equation shows that q_L is continuous in δ_2 and $q_L > 0$ holds if $\delta_2 = 0$. Thus, $q_L > 0$ holds for sufficiently small $\delta_2 > 0$. From $V_L = 1/v_L$ and (8), we have $r_L = q_L V_L$. □

Lemma A2 *Suppose that* $\sigma > 0$*,* $(1+\eta)(g+\delta_2) - \eta\mu > 0$*, and* δ_2 *is sufficiently small. Then, a unique bubbly steady-state equilibrium exists such that* $I_t > 0$ *holds, and* V^* *and* B^* *satisfy*

$$V^* = 1 - \sigma(\mu + \rho)^{\frac{1}{2}} \ (> 0), \tag{M.20}$$

$$B^* = \frac{\sigma(1 + \eta)(g + \delta_2)}{(\rho + \mu)^{\frac{1}{2}} + \eta\mu\sigma} - 1 \ (> 0), \tag{M.21}$$

if and only if

$$\underline{\sigma} \equiv \frac{(\rho + \mu)^{\frac{1}{2}}}{(1 + \eta)(g + \delta_2) - \eta\mu} < \sigma < \frac{1}{(\rho + \mu)^{\frac{1}{2}}}. \tag{M.22}$$

ζ^**,* k^**,* q^**, and* r^* *are given by* $\delta_1 \zeta^{*1+\frac{1}{\eta}} = [\rho + (1 - V^*)V^*/\sigma^2](1 + B^*)$*,* $f'(\zeta^* k^*) = \delta_1 \zeta^{*\frac{1}{\eta}}/V^*$*,* $q^* = [\zeta^* \delta'(\zeta^*) - \delta(\zeta^*)]/V^*$ (> 0)*,* $r^* = q^* V^*$*, respectively.*

(Proof) Suppose that $B_t > 0$. From $\dot{B}_t = 0$, (M.7), and (M.13), we have

$$\delta'(\zeta_t)\zeta_t = \left(\frac{1 - V_t}{\sigma^2} - \mu\right)(1 + B_t). \tag{M.23}$$

Substituting (M.13) and (M.23) into (M.6) yields $V = 1 - \sigma(\rho+\mu)^{1/2}$. Using $V_t = 1/v_t$ and (14), we rewrite (12c) as $I_t = (1-V_t)(1+B_t)K_t/\sigma^2$, meaning that $I_t > 0$ holds if $V_t < 1$. Thus, we obtain (M.20).

We set $\dot{k}_t = 0$ in (M.8) and have $(1-V_t)(1+B_t)/\sigma^2 = \delta(\zeta_t) + g$. From this equation, (M.6), (M.13), and the relation $\delta'(\zeta)\zeta = \delta_1 \zeta^{1+\frac{1}{\eta}} = \left(1+\frac{1}{\eta}\right)(\delta(\zeta) - \delta_2)$, we have

$$\rho + \frac{\left(1+\frac{1}{\eta}\right)(g+\delta_2)}{1+B} = \frac{1-V}{\sigma^2}\left(1+\frac{1}{\eta} - V\right). \tag{M.24}$$

Substituting (M.20) into (M.24) and solving for B_t yields (M.21).

Suppose that $V^* > 0$ and $B^* > 0$. Then, we have $\sigma < 1/(\rho+\mu)^{\frac{1}{2}}$ and $(\rho+\mu)^{\frac{1}{2}}/[(1+\eta)(g+\delta_2) - \eta\mu] < \sigma$, and thus the condition (M.22) holds. Conversely, suppose that (M.22) holds. The inequality $\sigma < 1/(\rho+\mu)^{\frac{1}{2}}$ implies $0 < V^* < 1$ and $I_t > 0$. The inequality $(\rho+\mu)^{1/2}/[(1+\eta)(g+\delta_2) - \eta\mu] < \sigma$ ensures $B^* > 0$.

With $B^* > 0$, ζ^*, k^*, q^*, and r^* are obtained in the same way in Lemma A1. □

We now prove Proposition A5. From (M.22), we have the following relationship:

$$\rho + \mu < (1+\eta)(g+\delta_2) - \eta\mu. \tag{M.25}$$

The inequality implies the following relationship:

$$\begin{aligned}
\frac{(\rho+\mu)^{1/2}}{(1+\eta)(g+\delta_2)-\eta\mu} &= \frac{(\rho+\mu)^{1/2}}{[(1+\eta)(g+\delta_2)-\eta\mu]^{1/2}} \frac{1}{[(1+\eta)(g+\delta_2)-\eta\mu]^{1/2}} \\
&< \frac{1}{[(1+\eta)(g+\delta_2)-\eta\mu]^{1/2}} \\
&< \frac{1}{\left(g+\delta_2+\frac{\eta}{1+\eta}\rho\right)^{1/2}} \\
&< \frac{1}{(\rho+\mu)^{1/2}}.
\end{aligned}$$

The second, third, and last lines use (M.25). Thus, we have:

$$\frac{(\rho+\mu)^{1/2}}{(1+\eta)(g+\delta_2)-\eta\mu} < \frac{1}{\left(g+\delta_2+\frac{\eta}{1+\eta}\rho\right)^{1/2}} < \frac{1}{(\rho+\mu)^{1/2}}. \tag{M.26}$$

From Lemma A1, A2, and (M.26), Proposition (A5) is proved. □

M.3 Proof of Proposition A6

Since the LHS of (M.24) decreases with B, (M.16) and (M.24) imply

$$\frac{1-V_L}{\sigma^2}\left(1+\frac{1}{\eta}-V_L\right) > \frac{1-V^*}{\sigma^2}\left(1+\frac{1}{\eta}-V^*\right). \tag{M.27}$$

Thus, we have $V^* > V_L$.

From (M.16) and (M.24), irrespective of whether bubbles exist or not, the following equation holds:

$$\frac{\rho(1+B) + \left(1+\frac{1}{\eta}\right)(g+\delta_2)}{\left(1+\frac{1}{\eta}-V\right)} = \frac{1-V}{\sigma^2}(1+B), \tag{M.28}$$

where $(V, B) = (V_L, 0)$ or $(V, B) = (V^*, B^*)$. The LHS of (M.28) increases with V and B. From $\dot{k}_t = 0$, $V^* > V_L$, $B^* > 0$, (M.8), and (M.28), we have the following relationship:

$$\delta(\zeta^*) + g = \frac{1-V^*}{\sigma^2}(1+B^*) > \frac{1-V_L}{\sigma^2} = \delta(\zeta_L) + g, \tag{M.29}$$

where $\delta(\zeta)$ increases with ζ. Thus, $\zeta^* > \zeta_L$ holds.

From $r = qV$ and (M.19), we rewrite r as:

$$r = \frac{1}{1+\eta}\delta_1\zeta^{1+\frac{1}{\eta}} - \delta_2, \tag{M.30}$$

where $(r, \zeta) = (r_L, \zeta_L)$ or $(r, \zeta) = (r^*, \zeta^*)$. From $\zeta^* > \zeta_L$, we have $r^* > r_L$.

We now prove (M.10). Suppose that σ is sufficiently close to $\overline{\sigma} = 1/\left[g+\delta_2+\left(\frac{\eta}{1+\eta}\right)\rho\right]^{\frac{1}{2}}$. Note that V_L is a positive solution of (M.16). By solving (M.16) for V, we have:

$$V_L = 1 - \frac{1}{2}\left\{-\frac{1}{\eta} + \sqrt{\frac{1}{\eta^2} + 4\sigma^2\left[\rho + \left(1+\frac{1}{\eta}\right)(g+\delta_2)\right]}\right\} \to 0 \quad \text{as} \quad \sigma \to \overline{\sigma}. \tag{M.31}$$

From (2) and (M.18), we have

$$\frac{\eta}{1+\eta}\delta_1\zeta_L^{1+\frac{1}{\eta}} + \delta_2 = \frac{1-V_L}{\sigma^2} - g \to \delta_2 + \frac{\eta}{1+\eta}\rho, \quad \text{as} \quad \sigma \to \overline{\sigma}. \tag{M.32}$$

Thus, we have:

$$\zeta_L \to \left(\frac{\rho}{\delta_1}\right)^{\frac{1}{1+\frac{1}{\eta}}} \equiv \overline{\zeta}_L \quad \text{as} \quad \sigma \to \overline{\sigma}. \tag{M.33}$$

Then, (M.13), (M.31), and (M.33) imply:

$$f'(\zeta_L k_L) = \frac{\delta'(\zeta_L)}{V_L} \to f'(\overline{\zeta}_L k_L) = \frac{\delta(\overline{\zeta}_L)}{0} = +\infty \quad \text{as} \quad \sigma \to \overline{\sigma}. \tag{M.34}$$

Since f satisfies the Inada condition, we have:

$$k_L \to 0, \quad \text{as} \quad \sigma \to \overline{\sigma}. \tag{M.35}$$

(M.35) implies that as σ converges to $\overline{\sigma}$, $y_L = Y_{L,t}/A_tL = f(\zeta_L k_L)$ also converges to zero. From $V_t = 1/v_t$, (12b), and (12c), we have:

$$c_L = \frac{C_{L,t}}{A_tL} = \frac{\rho}{V_L}k_L \quad \text{and} \quad i_L = \frac{I_{L,t}}{A_tL} = \frac{1-V_L}{\sigma^2}k_L. \tag{M.36}$$

From (M.35), c_L and i_L converge to zero, respectively. The market clearing condition $c_L^w = y_L - c_L - i_L$ implies that c_L^w also converge to zero.

The definition of $\overline{\sigma}$ and (M.26) imply:

$$\frac{(\rho+\mu)^{\frac{1}{2}}}{(1+\eta)(g+\delta_2)-\eta\mu} < \overline{\sigma} < \frac{1}{(\rho+\mu)^{\frac{1}{2}}}. \tag{M.37}$$

From Lemma A2, the bubbly steady state exists and we have $V^* > 0$, $\zeta^* > 0$, $k^* > 0$, $y^* > 0$, $c^* > 0$, $c^{w^*} > 0$, and $i^* > 0$ if $\sigma = \overline{\sigma}$, where c^* and i^* are given by:

$$c^* = \frac{C_t^*}{A_t L} = \frac{\rho}{V^*}(1+B^*)k^* > 0 \quad \text{and} \quad i^* = \frac{I_{L,t}}{A_t L} = \frac{1-V^*}{\sigma^2}(1+B^*)k^* > 0. \tag{M.38}$$

Thus, we have (M.10) if σ is sufficiently close to $\overline{\sigma}$. Proposition A6 is proved. □

M.4 Phase Diagram

To draw the phase diagram in Figure A2, we first demonstrate that V_t and ζ_t are functions of k_t and B_t. Given k_t and B_t, we plot the graphs of (M.6) and (M.13) in the (ζ_t, V_t) plane (see Figure A3). The LHS of (M.6) represents the marginal productivity of capital, which increases with ζ_t because ζ_t works like a capital-augmenting productivity. The RHS of (M.6) decreases with V_t. As a result, the graph of (M.6) is downward-sloping. The graph of (M.13) is upward-sloping because the term $\delta'(\zeta_t)/f'(\zeta_t k_t)$ increases with ζ_t. This occurs because $\delta''(\zeta_t) > 0$ and $f''(\zeta_t k_t) < 0$. Given k_t and B_t, the graphs of (M.6) and (M.13) intersect at a single point. Thus, V_t and ζ_t can be expressed as functions of k_t and B_t:

$$V_t = V(k_t, B_t) \qquad \text{and} \qquad \zeta_t = \zeta(k_t, B_t). \tag{M.39}$$

Figure A2 (a) shows that an increase in B_t shifts the graph of (M.6) to the right. As shown in Figure A2 (b), an increase in k_t shifts the graph of (M.6) to the right and the graph of (M.13) to the left. Thus, we have:

$$\frac{\partial V(k_t, B_t)}{\partial k_t} > 0, \quad \frac{\partial V(k_t, B_t)}{\partial B_t} > 0, \quad \text{and} \quad \frac{\partial \zeta(k_t, B_t)}{\partial B_t} > 0. \tag{M.40}$$

From (M.6) and (M.13), we have:

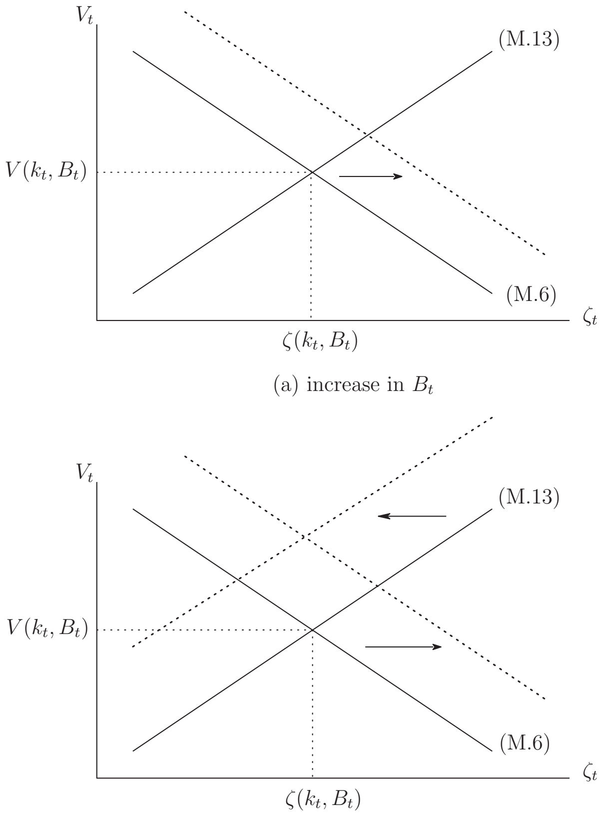
$$\frac{\partial \delta'(\zeta_t)\zeta_t}{\partial k_t} = \frac{1+B_t}{\sigma^2}(1-2V_t)\frac{\partial V_t}{\partial k_t} > (=)(<)0 \iff V_t < (=)(>)\frac{1}{2}. \tag{M.41}$$

Since $\delta'(\zeta_t)\zeta_t$ increases with ζ_t and $\partial V_t/\partial k_t > 0$, the following holds:

$$\frac{\partial \zeta_t}{\partial k_t} > (=)(<)0 \iff V_t < (=)(>)\frac{1}{2}. \tag{M.42}$$

We now derive the $\dot{B}_t = 0$ locus. By substituting (M.6) into (M.7), setting $\dot{B}_t = 0$ in (M.7), and solving for $V_t = V(k, B)$, we obtain:

$$V(k, B) = 1 - \sigma(\rho+\mu)^{\frac{1}{2}}. \tag{M.43}$$



(a) increase in B_t

(b) increase in k_t

Figure A3 $V(k_t, B_t)$ and $\zeta(k_t, B_t)$

The $\dot{B}_t = 0$ locus represents the combination of k and B that satisfies (M.43). Since $\partial V_t/\partial k_t > 0$ and $\partial V_t/\partial B_t > 0$, the $\dot{B}_t = 0$ locus is downward sloping. In the region below (above) the $\dot{B}_t = 0$ locus, we have $\dot{B}_t < 0$ ($\dot{B}_t > 0$), as shown in Figure A2.

Next, we derive the $\dot{k}_t = 0$ locus. Setting $\dot{k}_t = 0$ in (M.8) yields:

$$\frac{1 - V(k, B)}{\sigma^2} = \delta(\zeta(k, B)) + g. \tag{M.44}$$

The $\dot{k}_t = 0$ locus represents the combination of k and B that satisfies (M.44). Since the bubbly steady-state is unique and $k_L < k^*$ holds, we can plot the $\dot{k}_t = 0$ locus as shown in Figure A2. We assume that the partial derivative of (M.8) with respect to k_t around $(k_t, B_t) = (k_L, 0)$ is negative:

$$-\frac{1}{\sigma^2}\frac{\partial V(k_t, B_t)}{\partial k_t}\bigg|_{k_t=k_L, B_t=0} - \delta_1 \zeta_L^{\frac{1}{\eta}} \frac{\partial \zeta(k_t, B_t)}{\partial k_t}\bigg|_{k_t=k_L, B_t=0} < 0, \tag{M.45}$$

which ensures that $\dot{k}_t < 0$ around the bubbleless steady-state equilibrium. Then, in the region above (below) the $\dot{k}_t = 0$ locus, we have $\dot{k}_t > 0$ ($\dot{k}_t < 0$). As a result, the phase diagram in Figure A2 is obtained. The bubbly steady state is saddle-point stable, and the bubbleless steady state is stable. Consequently, after bubbles collapse, k_t converges to k_L.

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