DISCUSSION PAPER SERIES

Discussion paper No.278

Bubbly fundamentals

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September 2024

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Bubbly fundamentals *

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September 26, 2024

Abstract

Increases in the price-to-dividend ratio (PDR) have been observed during bubble periods. However, in the 2010s, asset prices have surged to bubble-era levels without a rise in the PDR. Based on this observation, we construct a macroeconomic model in which asset prices can be high or low under a constant PDR. In both equilibria, asset prices are entirely determined by the sum of expected future dividends and influence macroeconomic performance. The high asset price stimulates capital accumulation.

Keywords: asset bubbles, fundamental value, credit constraints, self-fulfilling expectation, multiple equilibria

JEL classification numbers: E44, G01, G12

*The authors are grateful to Tomohiro Hirano, Masaru Inaba, Ryo Jinnai, and Keiichi Kishi. This research was financially supported by the Japan Society for the Promotion of Science: Grant-in-Aid for Scientific Research (No. 19H01493, 19K01767, 20H05633, 21K01400, and 24K16341). All remaining errors are ours.

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1 Introduction

According the theory of rational bubbles, increases in stock prices are accompanied by an increase in the price-to-dividend ratio (PDR).[1] This theory is consistent with the past several episodes of asset price appreciation. During the asset bubble period in Japan, real stock prices and the PDR increased dramatically by 167% and 137%, respectively, from 1985 to 1989 (see the blue and orange lines in the gray area of Figure 1).[2] Similarly, in the U.S., stock prices and the PDR increased by 179% and 148%, respectively, from 1994 (just before the U.S. IT bubble began) to 1999 (see the blue and orange lines in the gray area of Figure 2).[3] The movement of the price-earnings-ratio (PER) is similar to that of the PDR (see appendix B).

However, the stock price increases during certain periods in Japan and the U.S., which are comparable to those in bubble periods, may not be consistent with rational bubble theory. In Japan, although stock prices increased by 203% between 2011 and 2023, the PDR only grew by 4.97% (see the blue and orange lines since 2011 in Figure 1).[4] Similarly in the U.S., the stock prices have been increasing significantly since 2009 (see Figure 2). In 2014, they eventually had surpassed the level of the IT bubble in 1999. However, the PDR didn't rise as much. During this period, while the stock prices increased by 140%, the PDR increase by only 10%. The stock price increases around 2020 were once again accompanied by that in the PDR. However, even extending the period to 2021 when the PDR temporarily increased, stock prices rose by 223%, while the PDR increased by only 56%. These stock price increases with the stable PDR may be inconsistent with rational bubble theory.

[1] See Appendix A for a more detail discussion.

[2] For Japanese data, nominal stock prices were obtained from the TOPIX, price indexes were obtained from the Consumer Price Index (National Composite), and average yields on the TSE First Section and Prime Market (dividend-paying companies) were taken from NIKKEI NEEDS-Financial QUEST. The CPI was seasonally adjusted using data from January 1980 to March 2024. Seasonal adjustment was used based on the assumption that there was a level change in the month in which the consumption tax was introduced and the month in which the consumption tax rate was raised. Real stock prices were calculated by dividing the TOPIX by the seasonally adjusted CPI, and the PDR was the inverse of the average yield. The average yield uses data from the TSE First Section until 2021, and data from the TSE Prime market after 2022. As in the U.S., real stock prices and the PDR at the end of each year are used as the real stock prices and the PDR for each year.

[3] For U.S. data, real S&P 500 stock price and real dividend data were obtained from the Schiller website, and TFP was obtained from The Penn World Table version 10.01. Since Schiller's data are monthly, the real S &P 500 stock price and real dividend in December of each year are used as the real S&P 500 stock price and real dividend for each year.

[4] Despite the significant rise in Japanese stock prices since 2011, the PDR has remained relatively stable, staying around 50. Even at its highest point in 2017, it was only about 69.

If an increase in asset prices is not a rational bubble, how can it be explained? Part of the rise in asset prices could be explained by productivity growth. The red line in Figure 1 shows the total factor productivity (TFP) in the Japan. The TFP growth is positive, however, it is quite gradual compared to the rise in stock prices. Indeed, the TFP increased by only 3.95% since 2011. Economic growth alone seems insufficient to explain the accelerated rise in stock prices since 2011. The TFP growth in the U.S. was also modest (see Figure 2). It increased by 5.76% from 2009 to 2019.[5]

Based on these observations, this study explores a theoretical explanation for whether asset prices can increase significantly even if the PDR remains constant and TFP growth is absent.[6] Moreover, we investigate the impacts of high asset prices on the economy and derive theoretical implications for asset pricing in an economy with a constant PDR.

Our model is based on the standard neoclassical economy with TFP normalized to one. There is no heterogeneity among economic agents. Households own firms as an asset. Thus, the market value of the firms represents the asset price in our model. The firms produce a single final good by using capital and labor. We introduce a timing mismatch between the payment for factor inputs and revenue from production. To deal with the timing mismatch, firms make a credit purchase contract. Credit constraints are then endogenously derived by an incentive-compatibility constraint of the firms. Due to the endogenous credit constraints, each firm's investment in capital is limited by a fraction of its market value.

This study yields the following five results that seem contradictory but are actually consistent with each other. First, the endogenous credit constraints yield multiple steady-state equilibria with high and low asset prices. Thus, our model exhibits a global indeterminacy. Second, the high stock price encourages capital accumulation. Third, in both equilibria with high and low asset prices, the asset price is entirely determined by the fundamental values, which is defined as the summation of the expected future dividends (or equivalently, firm's profits) stream. Fourth, the two steady states with different asset prices share the same PDR and TFP. Finally, even if high asset prices are not accompanied by a high PDR, the economy may still experience a sudden crash in asset prices. These results provide a theoretical

[5]During 1985-1989, the TFP in Japan increased by 6.3%. In the U.S., the TFP increased by 4% from 1994 to 1999.

[6]As long as the growth rate of TFP is constant, all of results in this study are unaffected.

foundation for the recent rises and crashes in stock prices without an increasing PDR.

The intuitions behind our results are as follows: The credit constraints create a positive feedback mechanism between asset prices and individual firms' investment. Suppose that the future profitability of a firm is expected to increase, which is reflected by an increase in the market value of the firm. The increased market value loosens the credit constraint and hence promotes the firms' investment. The increased investment leads to the actual increase in the future profits of the firm. Then, the (present) market value of the firm actually increases. The initial optimistic expectation about the future profitability is self-fulfilling. Thus, the equilibrium with high asset prices arises even if there are no fundamental changes such as technological changes.

We emphasize that the asset price appreciation in our model is not caused by speculation. Because no heterogeneity is included in our model, asset resale is excluded. Indeed, the asset price is determined by the discounted value of the future dividend stream. Our results suggest that even without changes in the PDR, asset prices could rise and fall significantly depending on the expectations about the future dividends (profits) stream and the rise and fall can have a significant impact on capital accumulation and macroeconomic performance. Since even the fundamental term of the firm value can vary depending on the expectations, we call it bubbly fundamental.

Related literature: Since the seminal contributions of Bernanke and Gertler (1989) and Kiyotaki and Moore (1997), many macroeconomic models have emphasized the importance of credit constraints. In the presence of credit constraints, relaxing them expands investment and production. This financial accelerator mechanism amplifies the effects of a fundamental shock such as a TFP shock, leading to significant macroeconomic fluctuations. In our model, without fundamental shocks, the feedback mechanism between asset prices and credit constraints is triggered by the self-fulfilling expectation.

The fact that the presence of credit constraints may be the source of equilibrium indeterminacy and self-fulfilling business cycles is not new. Examples include Benhabib and Wang (2013), Liu and Wang (2014), and Azariadis et al. (2016).[7] Assuming that the borrowing

[7] The seminal paper of Benhabib and Farmer (1994) show that an one-sector RBC model with increasing returns to scale can lead to indeterminacy of equilibria. Liu and Wang (2014) model with heterogeneous

capacity of firms depends on their output, Benhabib and Wang (2013) show that endogenous markups generate indeterminacy and self-fulfilling equilibria. In Liu and Wang (2014) and Azariadis et al.(2016), credit constraints affect the allocation of productive resources among heterogeneous firms and the aggregate productivity, which generates aggregate increasing returns and causes equilibrium indeterminacy. Unlike our study, none of these studies pay attention to the relationship between asset prices and the PDR.

Moreover, the source of indeterminacy in our model is different from these studies. Unlike Benhabib and Wang (2013), the credit limits of firms do not depend on their output. Unlike Liu and Wang (2014) and Azariadis et al.(2016), there is no heterogeneity in productivity. In our model, the expectation about future profitability of firms is self-fulfilling, which is the souce of indeterminacy.

Our work is also related to the rational bubble literature (Tirole (1985), Weil (1987), Fahri and Tirole (2011), Martin and Ventura (2012), Hirano and Yanagawa (2017), and Hori and Im (2023)). There are some similarities between rational bubble models and ours. Generally, both models have two types of steady-state equilibria with high and low asset prices.[8] However, the rational bubble models mainly focus on a purely bubble asset such as fiat money, of which fundamental value is always zero. Thus, these models cannot address the prices of dividend-yielding assets and indicators such as the PDR. In contrast, both of them can be addressed in our model.

Our model is closely related to the model of stock price bubbles in Miao and Wang (2018). Algthouh there are differences in the model setting between their and our models, the equilibrium dynamics in our model are exactly the same as that in Miao and Wang (2018). Thus, our model generates the same results as Miao and Wang (2018).

However, the implication for asset pricing is quite different. Miao and Wang (2018) interpret high stock prices as asset bubbles because the market value of a firm exceeds the value of capital it owns. Although their interpretation is plausible, it is different from the

firms is isomorphic to the Benhabib and Farmer (1994) model after aggregation. See Benhabib and Farmer (1999) for a survey of the literature.

[8] The local stability around the steady states in our model is the same as that in existing rational bubble models. In the models of rational bubbles, the bubbly steady state is determinate or unstable and the bubbeless steady state is stable. In our model, the steady state with a high stock price is locally determinate and the steady state with a low stock price is locally indeterminate.

definition of the rational bubbles as they themselves state (see footnote 1 of their paper). Hirano and Toda (2024) formally prove the nonexistence of rational bubbles in the model of Miao and Wang (2018). Besides, their complicated model makes it difficult to define the PDR. In contrast, our model is quite simple, which allows us to follow the rational bubble literature and to calculate the value of future dividend streams. We believe that our approach more directly connects the theoretical model to empirical data.

In spite of the same dynamic system as Miao and Wang (2018), we find that the asset price is determined by the fundamental term. We show that even fundamental term may rise and fall due to self-filling expectation, which has a significant impact on macroeconomic performance.

The rest of the paper is organized as follows. Section 2 presents our model. Section 3 characterizes the steady-state equilibrium with high and low asset prices. Section 4 derives asset price implications in our model. Concluding remarks are in Section 6.

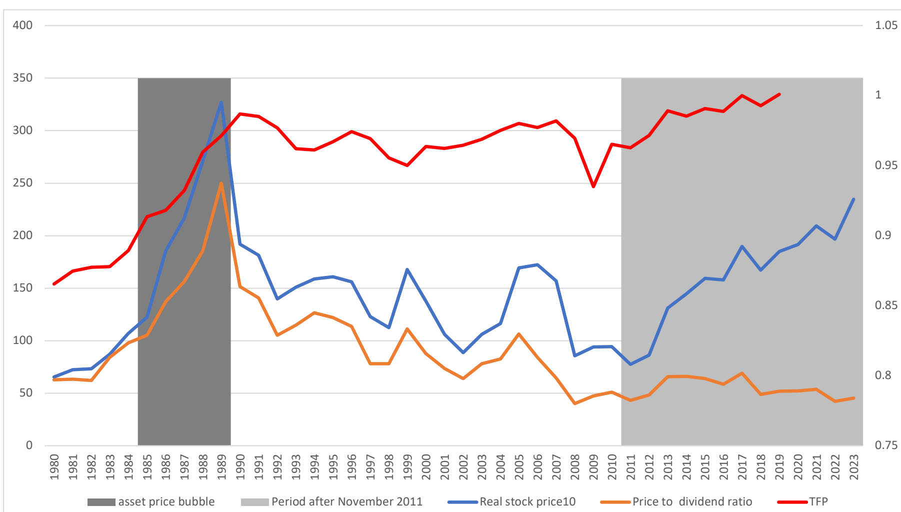

Figure 1

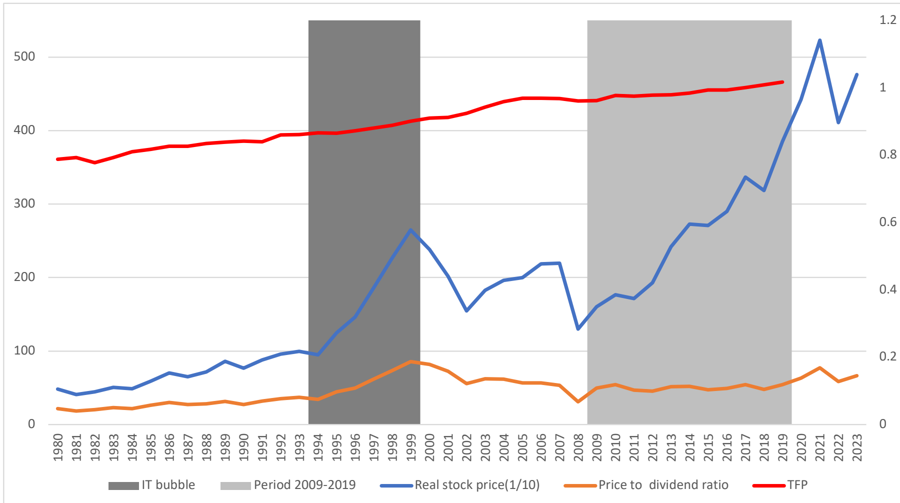

Figure 2

2 The Model with Endogenous Credit Constraints

Time is continuous and runs from $t = 0$ to ∞. All households are identical, so that we assume the representative household. There is a continuum of firms whose measure is one. Each firm is indexed by $j \in [0, 1]$. The market value of firm j is V_t^j and it delivers dividend D_t^j at time t. There are no aggregate and idiosyncratic uncertainty in the economy.

2.1 Households

As in Miao and Wang (2018), the representative household is endowed with a linear utility, $U_t = \int_t^\infty c_s e^{-r(s-t)} ds$, where $r > 0$ is a (constant) subjective discount rate and c_t is consumption at time t. The representative household inelastically supplies one unit of labor and owns the firms through stock holdings. Its budget constraint is $\int V_t^j \dot{\phi}_t^j dj + c_t = w_t + \int D_t^j \phi_t^j dj$, where w_t is the wage rate, ϕ_t^j denotes the holdings of firm j's stock, and thus $\dot{\phi}_t^j$ denotes the purchasing of firm j's stock. The utility maximization yields

$$rV_t^j = D_t^j + \dot{V}_t^j, \tag{1}$$

and the transversality condition is

$$\lim_{T\to\infty} V_T^j \phi_T^j e^{-rT} = \lim_{T\to\infty} V_T^j e^{-rT} = 0. \tag{2}$$

In (2), we use the stock market clearing condition $\phi_t^j = 1$.

For simplicity, we assume the linear utility by following Miao and Wang (2018). However, a concave utility function does not change our main result. The point is that the interest rate becomes equal to the subjective discount rate at steady state(s). See the discussion just below Proposition 5.

2.2 Firms

Each firm produces a single final good that can be used for consumption and capital accumulation. We normalize the price of the final good to one. The production function is a standard neoclassical production function $Y_t^j = F(K_t^j, N_t^j)$, where K_t^j and N_t^j are capital and labor inputs of firm j, respectively. $F(K_t^j, N_t^j)$ is constant returns to scale in capital and labor, exhibits positive and diminishing marginal products in both arguments, and satisfies the Inada condition and $F(0, N_t^j) = F(K_t^j, 0) = 0$.

We describe firm j's behavior in the infinitesimally short time interval between time t and $t + dt$. Here, t and dt can differ among firms because different firms could have different planning periods.[9] At time t, firm j owns the capital stock K_t^j. Unlike Miao and Wang (2018), there is no trading of capital K_t^j among firms. This means that capital of firm j is specific to the firm. For example, K_t^j may reflect the production technology and productivity specific to firm j. In the time interval $[t, t + dt]$, firm j employs $N_t^j dt$ units of labor. At the end of this interval (time $t + dt$), firm j completes its final good production and sells it to households as a consumption good and other firms as an investment good, earning the revenue $Y_t^j dt$.

At the same time, firm j invests $I_t^j dt$ units of the final good produced by other firms in the capital stock during the interval of $[t, t+dt]$. We assume that firm j's own output cannot

[9]Putting firm index j on the time interval, as in $[t^j, t^j + dt^j]$, might be more precise. However, to make the notation simple, we omit index j from the time interval.

be used for firm j's investment. Then, K_t^j evolves according to

$$dK_t^j = (I_t^j - \delta K_t^j)dt, \tag{3}$$

where $\delta > 0$ is the capital depreciation rate.

2.3 Endogenous Credit Constraints

Note that there is a timing mismatch. Firm j invests in capital at the start of the time interval (time t), while it earns revenue from production at the end of the time interval (time $t + dt$). To finance its investment, firm j makes a credit purchase contract with other firms. Firm j procures $L_t^j dt$ units of the final good at time t with a commitment to repay it at time $t + dt$ after it earns revenue.[10] No interest is charged on this payment. Assume that $\eta \in [0, 1]$ fraction of investment $I_t^j dt$ must be financed by the credit purchase contract. Firm j must satisfy

$$\eta I_t^j dt \leq L_t^j dt. \tag{4}$$

As in Miao and Wang (2018), the wage payment $w_t N_t^j$ is not subject to timing mismatch.[11,12]

The enforcement of credit contracts is imperfect, so that firm j has an option of default. Let $v^N(K_t^j, t)$ and $v^D(K_t^j, t)$ be the stock values of a non-defaulting and defaulting firm with capital stock K_t^j at time t, respectively. We assume that $v^N(K_t^j, t)$ and $v^D(K_t^j, t)$ are differentiable with respect to both arguments. Define $v(K_t^j, t) = \max\left\{v^N(K_t^j, t), v^D(K_t^j, t)\right\}$.

A defaulting firm does not repay $L_t^j dt$. It is detected with probability ζdt during the interval of $[t, t + dt]$. If detected, it is forced to pay a penalty depending on its market value. Assume that the penalty payment is equal to $v\left(\lambda(K_t^j + dK_t^j), t + dt\right)$, where $\lambda > 0$. The

[10] At time $t + dt$, firm j repays by using sales of consumption goods to households, the $1 - \eta$ fraction of investment goods sales to other firms, and repayments from other firms.

[11] In Benhabib and Wang (2013) and Liu and Wang (2014), the wage and interest payments to households, $w_t N_t$ and $r_t K_t$, are subject to timing mismatch. In their models, households wait to receive income from labor and capital, so that they effectively provide credit to firms. Thus, firms make a credit contract with households. In contrast, we focus on the credit contract among firms.

[12] Our main results are not affected even if the wage payment $w_t N_t^j$ is subject to timing mismatch.

detected firm continues its operation after the penalty payment. Hence, its value satisfies

$$
\begin{aligned}
v^D(K_t^j, t) =& (Y_t^j - w_t N_t^j - I_t^j + L_t^j)dt + \frac{1-\zeta dt}{1+rdt} v\left(K_t^j + dK_t^j, t+dt\right) \\
&+ \frac{\zeta dt}{1+rdt}\left[v\left(K_t^j + dK_t^j, t+dt\right) - v\left(\lambda(K_t^j + dK_t^j), t+dt\right)\right]. \qquad (5)
\end{aligned}
$$

As a non-defaulting firm repays $L_t^j dt$, its value satisfies

$$
v^N(K_t^j, t) = (Y_t^j - w_t N_t^j - I_t^j)dt + \frac{1}{1+rdt} v\left(K_t^j + dK_t^j, t+dt\right). \qquad (6)
$$

We impose an incentive-compatibility constraint for the firm j, $v^N(K_t^j, t) \geq v^D(K_t^j, t)$. If $v^N(K_t^j, t) \geq v^D(K_t^j, t)$, firms have no incentive to default. Using (4), (5), (6), and the approximation of $dt \cdot dt = 0$ and taking $dt \to 0$, we can rewrite this inequality as

$$
\eta I_t^j \leq \zeta v\left(\lambda K_t^j, t\right). \qquad (7)
$$

We simply call (7) the *credit constraint*. This inequality states that firm j's investment is limited by a fraction of its market value. We interpret ζ/η in (7) as the extent of the credit constraints, which is common to all firms. The increase in ζ/η stimulates firms' investment.

2.4 Optimization and First-order Conditions

As long as (7) is satisfied, we have $v(K_t^j, t) = v^N(K_t^j, t)$. We rearrange (6) and take the limit $dt \to 0$ to obtain the following maximization problem of firm j:

$$
rv(K_t^j, t) = \max_{N_t^j, I_t^j} (Y_t^j - w_t N_t^j - I_t^j) + \frac{\partial v\left(K_t^j, t\right)}{\partial K}(I_t^j - \delta K_t^j) + \frac{\partial v\left(K_t^j, t\right)}{\partial t}, \text{ s.t. (7).} \qquad (8)
$$

The first-order conditions of the problem are given by

$$
w_t = \frac{\partial F(K_t^j, N_t^j)}{\partial N}, \qquad (9)
$$

$$
\eta \mu_t = \frac{\partial v\left(K_t^j, t\right)}{\partial K} - 1, \qquad (10)
$$

where μ_t is the Lagrangian-multiplier associated with (7). In (10), $\partial v\left(K_t^j, t\right)/\partial K$ is Tobin's marginal q. If and only if the credit constraint (7) binds, Tobin's marginal q is greater than investment cost (=1). Using (9), we obtain

$$Y_t^j - w_t N_t^j = \frac{\partial F(K_t^j, N_t^j)}{\partial K_t^j} K_t^j \equiv R_t K_t^j. \tag{11}$$

3 Equilibrium Dynamics and Steady State

This section derives equilibrium dynamics and the steady state equilibrium. We just present our results here. The next section discusses the mechanisms and implications of our results.

3.1 Equilibrium Dynamics

As in Miao and Wang (2018), we conjecture that $v(K_t, t)$ is linear in K_t^j as follows:

$$v(K^j, t) = Q_t K^j + Z_t. \tag{12}$$

In (12), we intentionally drop time index t from K^j to emphasize that the functional form of $v(K^j, t)$ depends on time t, which is reflected by the time dependence of Q_t and Z_t. Then, we have

$$\frac{\partial v(K^j, t)}{\partial t} = \dot{Q}_t K^j + \dot{Z}_t \text{ and } \frac{\partial v(K^j, t)}{\partial K^j} = Q_t. \tag{13}$$

Because the first equation shows how the functional form changes over time, we need not differentiate one of the arguments, K_t^j, with respect to t.

The following Proposition characterizes the dynamics of Z_t and Q_t.

Proposition 1 *Suppose that $Q_t > 1$. Then, (7) binds and we have*

$$I_t^j = \frac{\zeta}{\eta} \left(\lambda Q_t K_t^j + Z_t \right). \tag{14}$$

Z_t and Q_t satisfy the following differential equations:

$$\dot{Z}_t = \left\{ r - \frac{\zeta}{\eta}(Q_t - 1) \right\} Z_t, \tag{15}$$

$$\dot{Q}_t = (r + \delta)Q_t - R_t - \frac{\zeta}{\eta}\lambda(Q_t - 1)Q_t, \tag{16}$$

as well as the transversality conditions

$$\lim_{T\to\infty} Q_T K_T^j e^{-rT} = \lim_{T\to\infty} Z_T e^{-rT} = 0. \tag{17}$$

(Proof) See Appendix C.

Equation (14) implies that large ζ and λ mean that the default costs are large and hence firms have less incentive to default. A small η suggests that investment is less subject to the credit constraint. Thus, large ζ/η and λ indicate a loose credit constraint in our model.

3.2 Steady State Equilibria

If $\eta > 0$, the equilibrium dynamics have two steady states: one with $Z_t = 0$ and the other with $Z_t > 0$. For simplicity, we focus on symmetric equilibria where $K_t^j = K_t$ for all j and omit firm index j. When we focus on the steady state, we omit time index t from variables that are constant in a steady state. Besides, in the remainder of the paper, the variables with an asterisk, such as Q^*, K^*, and Z^*, represent the steady-state variables with $Z_t > 0$. The firm value with K^* is denoted by $v^*(K^*)$. Variables and values with double asterisks, such as Q^{**}, K^{**}, Z^{**}, and $v^{**}(K^{**})$, denote the steady-state ones in the steady-state equilibrium with $Z = 0$. The following proposition shows the existence of the steady-state equilibria

Proposition 2 *Suppose that $\eta > 0$. (i) There exists a unique steady-state equilibrium with $Z^{**} = 0$ in which (7) binds and Q^{**}, R^{**} and K^{**} satisfy*

$$Q^{**} = \frac{\eta}{\zeta}\frac{\delta}{\lambda}(> 1), \quad R^{**} = Q^{**}r + \delta, \quad \textit{and} \quad \frac{\partial F(K^{**}, 1)}{\partial K} = R^{**}, \tag{18a}$$

if and only if

$$0 < \lambda < \frac{\eta}{\zeta}\delta. \tag{18b}$$

(ii) There exists a unique steady-state equilibrium in which (7) *binds and* Z^**,* Q^**,* R^**, and* K^* *satisfy*

$$\frac{Z^*}{K^*} = \frac{\eta}{\zeta}\delta - \lambda\left(1 + \frac{\eta}{\zeta}r\right) > 0, \tag{18c}$$

$$Q^* = 1 + \frac{\eta}{\zeta}r(> 1), \quad R^* = \left[(1-\lambda)r + \delta\right] Q^*, \quad \text{and} \quad \frac{\partial F(K^*, 1)}{\partial K} = R^*, \tag{18d}$$

if and only if

$$0 < \lambda < \frac{\delta}{\frac{\zeta}{\eta} + r}. \tag{18e}$$

(Proof) See Appendix D.

Because (18e) implies (18b), the two steady states coexist. If λ is small and η/ζ is large to satisfy (18e), then the credit constraints bind in the both steady states.

3.3 Relationship to Miao and Wang (2018)

Propositions 1 and 2 are analogous to Propositions 1, 3, and 4 in Miao and Wang (2018). Set $\frac{\zeta}{\eta} \equiv \pi < 1$ in (14), (15), and (16). Then, the dynamic system in our model becomes the same as that of Miao and Wang (2018). If we set $\frac{\zeta}{\eta} = \pi$ in all equations and inequalities of Proposition 2, then, all steady-state values and the existence conditions are identical to those presented in Miao and Wang (2018) (see Propositions 3 and 4 in Miao and Wang (2018)).

3.4 Local Stability around the Steady States

Because the dynamics system of our model is the same as that of Miao and Wang (2018), our model has the same local dynamics around the steady states as those in Miao and Wang (2018).[13] The steady states with $Z^*(> 0)$ is locally determinate. For any $K_0 > 0$, there exists a unique equilibrium with $Z_t > 0$ converging to the steady state with $Z^*(> 0)$ as t approaches ∞. The steady state with $Z = 0$ is locally indeterminate. Given K_0, there exist

[13] See Online Appendix A in Miao and Wang (2018) for the formal prove.

infinitely many equilibria starting $Z_0 \geq 0$ such that these equilibrium paths converge to the steady state with $Z = 0$ as t approaches ∞.

Although our model provides the same equilibrium characteristics as Miao and Wang (2018)'s model, there are some important differences between their and our models. First, firms in their model are subject to idiosyncratic shocks, whereas firms in our model are free from uncertainty. Second, capital is traded at a price of Q_t in their model. We assumes that capital of a firm is specific to the firm. Hence, capital is not traded in the market.

3.5 Comparison between the Two Steady States

Since inequality (18e) implies (18b), the following Proposition is obtained.

Proposition 3 *Assume that inequality* (18e) *holds. Then, there exist two steady-state equilibria.*
(i) Capital stock: We have

$$K^{**} < K^{*} \quad \text{and} \quad v^{**}(\lambda K^{**}) < v^{*}(\lambda K^{*}).$$

(ii) Market value: Assume that $F(K_t, N_t) = K^{\alpha} N^{1-\alpha}$ ($\alpha \in (0,1)$). If $\lambda > 0$ is not too large, we have

$$v^{**}(K^{**}) < Q^{*}K^{*} < v^{*}(K^{*}).$$

(Proof) See Appendix E.

Proposition 3 shows that under the same parameter set, multiple steady states with high and low market values arise in the economy. Moreover, Proposition 3 simply states that the high market value is associated with large capital stock level. The mechanism behind Proposition 3 is discussed in the next section.

4 Asset Price, Fundamental Value, and Bubbles

As discussed in Section 3.4, our model provides the same results as the model of Miao and Wang (2018). Assuming that K_t is traded at a price of Q_t in the market, they interpret that $Q_t K_t$ is the fundamental value of a firm and Z_t represents an asset bubble.[14] Although their interpretation is plausible and intuitive, it is different from the definition commonly used in the *rational* bubble literature, as they themselves point out in footnote 1 of their paper. Moreover, we assume that K_t^j is specific to firm j and is not traded in the market. Thus, the direct application of Miao and Wang (2018)'s interpretaion of Z_t to our model may be problematic. Because the purpose of the present study is to examine the PDR, we follow the rational bubble literature by examining the difference between the asset price and the fundamental value. We believe that our approach more directly connects the theoretical model to empirical data.

4.1 Asset Price and PDR

In the utility maximization problem of the household, we denote the stock price of a firm as V_t. As $V_t = v(K_t, t)$ holds in equilibrium, we have

$$\dot{V}_t = \frac{\partial v(K_t, t)}{\partial K}\frac{dK_t}{dt} + \frac{\partial v(K_t, t)}{\partial t} = \frac{\partial v(K_t, t)}{\partial K}(I_t - \delta K_t) + \frac{\partial v(K_t, t)}{\partial t}. \tag{19}$$

By using (3), (11), (13), and (19), we rewrite (8) as

$$rV_t = R_t K_t - I_t + \dot{V}_t. \tag{20}$$

We have derived the above equation by solving the firm's optimization problem. This equation must be consistent with the no-arbitrage condition (1) that is derived from the households' optimization problem. Comparison between (1) and (20) yields

$$D_t = R_t K_t - I_t.$$

[14]Under this interpretation, agents can profit from the following arbitrage trade. An agent purchases K_t units of capital at a cost of $Q_t K_t$. After that, He or she starts up a business and then sells the business at a price of $Q_t K_t + Z_t$. Miao and Wang (2018) exclude such an arbitrage trade by assumption.

The dividend of a firm is equal to the operating profits minus investment and is distributed to households. By solving (20), we derive the familiar asset price equation:

$$V_t = \underbrace{\int_t^\infty (R_v K_v - I_v)\, e^{-r(v-t)} dv}_{V_t^f} \quad + \quad \underbrace{\lim_{T\to\infty} V_T e^{-r(T-t)}}_{B_t}.$$

According to the familiar definition, the term V_t^f is the fundamental term, which is the present value of the future dividend sequence $\lim_{T\to\infty} \int_0^T D_t e^{-\int_0^t r_s ds} dt$. The term B_t is the bubble term.

Focusing on the steady states, we evaluate V^f and measure the deviation of the asset price from the fundamental term in the following proposition.

Proposition 4 *At the steady states, v^f is given by*

$$V^f \equiv \int_t^\infty (R_v K_v - I_v)\, e^{-r(v-t)} dv = QK + Z \equiv V, \tag{21}$$

where $(V^f, V, Q, K, Z) = ({V^f}^*, V^*, Q^*, K^*, Z^*)$ *or* $({V^f}^{**}, V^{**}, Q^{**}, K^{**}, 0)$.

(Proof) See Appendix F.

Clearly, (21) implies $B_t = 0$. The stock price is entirely determined by the sum of the discounted future dividend and the bubble term is zero. Moreover, the bubble term is zero even along the transitional dynamics. As Santos and Woodford (1997) point out (see equation (2.4) in their paper), bubble can never start in rational bubble models. Thus, if the bubble term is zero in a steady state, it is also zero along a transitional path converging to it.[15]

Our result is consistent with the (non-)existence conditions of *rational* bubbles in the literature. Many authors repeatedly show that rational bubbles exist if and only if the economic growth rate exceeds the interest rate in the economy without bubbles. In our

[15]Because V_t^f is a solution of (20), we have $rV_t^f = D_t + \dot{V}_t^f$. Thus, the bubble term $B_t \equiv V_t - V_t^f$ follows $\dot{B}_t = rB_t$. If $B_t = 0$ holds for some t, it must hold for other ts.

model, the economy is not growing while the interest (discount) rate is strictly positive.[16,17] Besides, Hirano and Toda (2024) formally prove the nonexistence of rational bubbles in the model of Miao and Wang (2018).

We next derive the PDRs in the two steady states. We now know that the stock price is entirely determined by the sum of the discounted future dividend. Thus, in the two steady states, we have

$$V = \frac{D}{r},$$

where $(V, D) = (V^*, D^*)$ or (V^{**}, D^{**}). We obtain the following proposition.

Proposition 5 *The two steady stats have the same PDR:*

$$PDR^* = \frac{1}{r} = PDR^{**}.$$

Note that even if we employ a concave utility function, Proposition 5 holds. This is because the interest rate becomes equal to the subjective discount rate of the household at the steady states.

4.2 Implications and Interpretations

Propositions 3 (ii) imply that the market price of a firm is not uniquely determined in our model. On the other hand, Proposition 5 shows that the PDR is uniquely determined. Different steady states with the same PDR have different the firm values. Our model shows that even without significant changes in the PDR, the stock price of the firms can change sharply. This may happen if productivity of the economy is different in the different steady states. However, this is not the case in our model. Because (18e) implies (18b), the two

[16] In addition, Kocherlakota (1992) shows that in any equilibrium with asset bubbles, the discounted value of the aggregate endowment is infinite. Similarly, Santos and Woodford (1997) prove that asset bubbles do not exist if the net supply of the asset is positive and the discounted value of the aggregate endowment is finite. Our non-existence result of bubbles is consistent with these two influential studies.

[17] Rational bubble models usually assume some forms of heterogeneity to ensure that arbitrage trading of assets takes place among economic agents. Many authors use overlapping-generations frameworks or introduce idiosyncratic shocks to generate heterogeneity. See Hori and Im (2023) for example. In our model, both households and firms are homogeneous.

steady states coexist under the same set of parameters. This means that the two steady states share a common productivity.

In sum, the two steady states have the same PDR and the same TFP. Nevertheless, they have the different asset prices. This is consistent with the recent trend observed in the United States and Japan.

We emphasize that the high asset price is not caused by speculation. Because no heterogeneity is included in our mode, asset resale is excluded. To understand the mechanism behind the multiple equilibria and the high asset price, we replicate the credit constraint of firm j, (7), assuming that it is binding:

$$\eta I_t^j = \zeta v^j(\lambda K_t^j, t).$$

Remember that the market value of the firm is entirely determined by the future dividend stream, given the current capital stock level. The dividend is equal to the operating profits net of investment costs. Thus, $v^j(\lambda K_t^j, t)$ represents a discounted value of the future profit stream that starts from λK_t^j. Suppose that the future profits of firm j is expected to increase. This expectation immediately increases $v^j(\lambda K_t^j, t)$. As a result, the credit constraint of firm j is eased, allowing firm j to expand investments and accumulate a greater amount of capital. Consequently, the future profit stream indeed increases. The initial optimistic expectation about profitability becomes self-fulfilling. This self-fulfilling expectation leads to the existence of multiple equilibria, where the market value of the firm can be high or low. Since the future dividend stream depends on the expectation of economic agents in our model, we call V^f the *bubbly* fundamental value, instead of calling it the fundamental value simply.[18]

5 Bursting Asset Price

In this section, we examine whether a stock price surge without an increase in the PDR is likely to be sustained perpetually. Specifically, we investigate whether a sunspot shock could

[18]Even if the wage payment is subject to timing mismatch, main results are still hold. Even in this environment, there exist multiple steady state with $Z^* > 0$ and $Z^{**} = 0$, respectively. In both steady states, asset prices are entirely determined by the sum of the discounted dividend. In equilibrium, the fundamental term is determined by the self-fulfilling expectation

cause a sudden decline in the stock value. If such a sunspot equilibrium does not exist, a stock price rise without the rising PDR is sustainable. However, if it exists, even without notable increases in the PDR, stock prices may suffer a sudden crash.

Remember that the stock price of a firm is given by $V_t = v(K_t, t) = Q_t K_t + Z_t$ (see (12)) in equilibrium. We focus on a sunspot shock that changes the term Z_t from a positive value to zero. We assume that the economy is initially on a steady state with $Z_t > 0$. Given $Z_t > 0$, $Z_{t+dt} > 0$ continues to hold with probability $1 - \varepsilon dt$ ($\varepsilon > 0$). Otherwise, Z_{t+dt} becomes equal to zero. Once $Z_{t+dt} = 0$ is realized, we assume that $Z_t = 0$ holds in subsequent future. Then, the economy converges to the steady state with $Z^{**} = 0$, which is characterized by the case (i) in Proposition 2. The sunspot shock is independent of the event of detecting the defaulting firm. Appendix G provides the derivation of the Bellman equation and the equilibrium dynamics.

In the presence of the sunspot shock, the initial steady state is generally different from the steady state with Z^*. We call it the sunspot steady-state equilibrium. The following proposition shows the existence of the sunspot equilibrium.

Proposition 6 *Suppose that* $\eta > 0$. *If and only if*

$$0 < \lambda < \frac{\delta}{\frac{\zeta}{\eta} + r + \varepsilon} \tag{22}$$

holds, then there exists a unique sunspot steady-state equilibrium.

(Proof) See Appendix G.1.

Thus, the stock price of the firms may suffer a sudden clash even if their PDRs do not show significant changes.

Finally, we numerically show that a sunspot shock actually causes a sudden decline in asset prices under some parameters condition.[19] See Figure 3. The economy is initially in the sunspot steady-state equilibrium, and then the sunspot shock hits the economy at time $t = 0$. With the realization of the shock, the stock price of the firms suddenly decline.

[19]We specify the production function as the standard Cobb-Douglass production function, $F(K_t, N_t) = K_t^{\alpha} N_t^{1-\alpha}$ where $0 < \alpha < 1$. We set $\alpha = 0.4$, $r = 0.4$, $\delta = 0.25$, $\eta = 1$, $\zeta = 0.01$ ($\zeta/\eta = \pi = 0.01$), $\lambda = 0.2$, and $\varepsilon = 0.05$. All of these parameters are exactly same as those used in Miao and Wang (2018).

After the sudden crash, the stock price and capital stock gradually move to the steady-state equilibrium with $Z^{**} = 0$. Thus, the economy suffers a sudden decline in the market price of the firms, followed by a subsequent recession, even if there are no increases in the PDR and no negative shocks to the fundamentals.

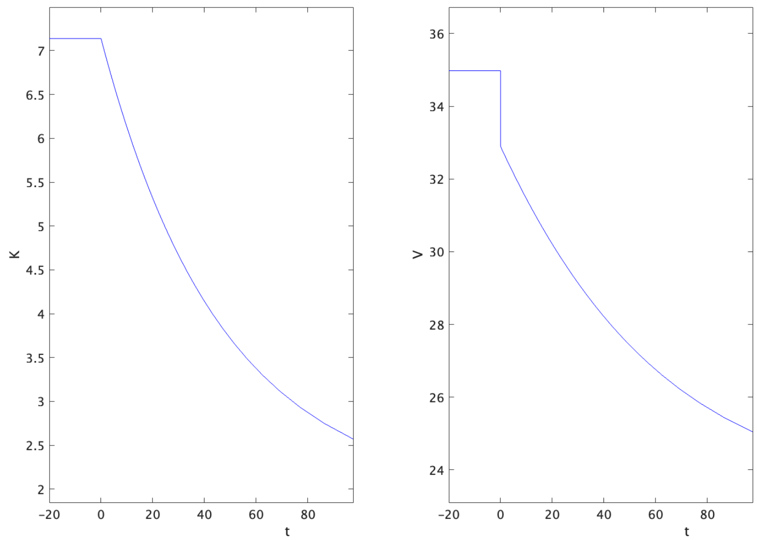

Figure 3 Sudden Clash in Stock Prices

6 Conclusion

We constructed a model with endogenous credit constraints and no TFP growth. Two types of steady-state equilibria are obtained. The two steady states have the same PDR and the same TFP, nevertheless, stock prices can be high or low. In both equilibria, stock prices are entirely determined by the sum of the future dividend stream. The future dividend stream depends on the expectation of agents. Thus, the self-fulfilling expectation determines stock prices and affects macroeconomic performances. The high stock price significantly stimulates capital accumulation.

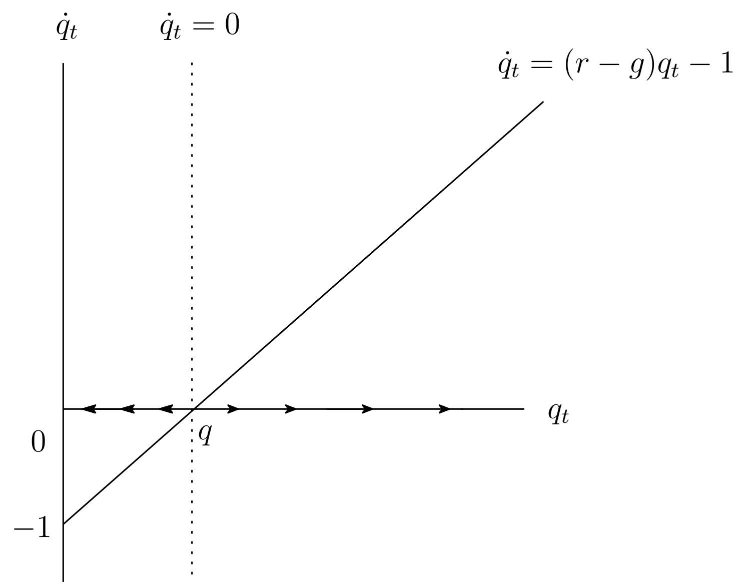

Figure A1 Dynamics of q_t

Appendix

A Rational Bubbles and the PDR

Rational bubbles are defined as the difference between the market price of an asset and its fundamental value. The latter is defined by the summation of the present value of the future dividend stream. Consider the standard no-arbitrage condition: $rV_t = \dot{V}_t + D_t$, where r is the market interest rate, V_t is the asset price, and D_t is the dividend of the asset. Define the price-to-dividend ratio (PDR) by $q_t \equiv V_t/D_t$. Then, q_t satisfies $\dot{q}_t = (r-g)q_t - 1$, where g is the growth rate of the dividend. If the PDR is constant ($\dot{q}_t = 0$), $q = 1/(r-g)$ and equivalently $V_t = D_t/(r-g)$ hold (see Figure A1). Thus, the asset price is equal to the fundamental value. On the other hand, if there is a rational bubble ($q_t > q$, or equivalently, $V_t > D_t/(r-g)$), the PDR monotonically increases over time.

B PER

The PER also behaves similarly to the PDR during certain periods. In Japan, the PER (Shiller10) increased by 123% from 1985 to 1989, which is comparable to a 167% increase in stock prices. In contrast, the PER decreased by 27% from 2011 to 2023 although stock price increased by 203% during the same period. In the U.S., the PER (Shiller10) increased by 122% from 1994 to 1999 and stock prices increased by 179%. The PER increased by 49% from 2009 to 2019. This is larger than the increase in the PDR (10%). Most notably, however, it is much smaller than the 140% rise in stock prices.

The Shiller PER is calculated based on the average real earnings per share over the past 10 years. The PER (Shiller10) is based on 10-year average earnings per share. In Japan, monthly earnings per share data are available only from June 1980. Since the Japanese PER (Shiller10) can only be calculated from May 1990, we use the five-year average of earnings per share to calculate the PER(Shiller5) from May 1985 to December 1994.[20,21]

[20] The U.S. Shiller PER is obtained from the Shiller website. The Japanese Shiller PER is calculated from earnings per share and the average stock price. Earnings per share are obtained from Tokyo Stock Exchange, Inc. The average stock price is the Arithmetic Stock Price Average obtained from NIKKEI NEEDS-Financial QUEST and Tokyo Stock Exchange, Inc. Earnings per share and average share price use data from the TSE First Section until March 2022, and data from the TSE Prime Market from April 2022 onward. As of September 24, 2015, the method of calculating the average stock price was changed, so the value of the average stock price prior to September 2015 would be 10 times the value when calculated using the new method. In determining the PER, the value of the average stock price prior to September 2015 is multiplied by 10, and the value calculated using the new method is used as is from October 2015. The TSE publishes two average stock prices for September 2015, one before and one after September 24, but data after September 24 were used. Average stock prices for other months were obtained from Nikkei Needs. The earnings per share calculation method was changed in October 2018, and if data prior to September 2018 is calculated using the new method, it will be 10 times the published value. For data prior to September 2018, we use 10 times the published value.

[21] The correlation coefficient between PER(Shiller10) and PER(Shiller5) from May 1990 to December 1994 is 0.993. if PER(Shiller10) can be calculated from May 1985 to April 1990, PER(Shiller10) is Shiller5), then it is likely that the PER(Shiller10) would have been similar to the PER(Shiller5).

C Proof of Proposition 1

Suppose that $Q_t > 1$. Then, (7) binds and we must have (14). By using (11), (12), (13), and (14), we rewrite (8) as

$$
\begin{aligned}
r(Q_t K_t^j + Z_t) &= R_t K_t^j - I_t^j + Q_t(I_t^j - \delta K_t^j) + \dot{Q}_t K_t^j + \dot{Z}_t \\
&= R_t K_t^j - \delta Q_t K_t^j + (Q_t - 1) I_t^j + \dot{Q}_t K_t^j + \dot{Z}_t \\
&= R_t K_t^j - \delta Q_t K_t^j + (Q_t - 1)\frac{\zeta}{\eta}\left(\lambda Q_t K_t^j + Z_t\right) + \dot{Q}_t K_t^j + \dot{Z}_t. \qquad \text{(C.1)}
\end{aligned}
$$

As the relationship (C.1) holds for any $K_t^j > 0$, then (15) and (16) must hold. Further as $V_t^j = v(K_t^j, t)$ holds in equilibrium, (2) ensures (17). □

D Proof of Proposition 2

Proof of (i): Suppose that $Q_t > 1$. Then, (14) holds from Proposition 1. Using $\dot{K}_t = 0$ and $K_t^j = K_t$, we rewrite (3) as $I = \delta K$. Substituting $I = \delta K$ into (14) yields Q^{**} in (18a). Set $\dot{Q}_t = 0$ in (16). Substituting Q^{**} into (16) yields R^{**} in (18a). Obviously, Q^{**} and R^{**} are uniquely determined by (18a). Moreover, Q^{**} in (18a) shows that $Q^{**} > 1$ holds if and only if (18b). Equation (11) shows that K^{**} satisfies $R^{**} = \frac{\partial F(K^{**},1)}{\partial K}$. The Inada condition ensures the uniqueness of K^{**} since $R^{**} > 0$. □

Proof of (ii): Setting $\dot{Z}_t = 0$ into (15) yields Q^*. Obviously, $Q^* > 1$ holds, meaning that the credit constrains bind, and thus (14) holds. (14) and $I = \delta K$ yields

$$\left(\delta - \frac{\zeta}{\eta}\lambda Q^*\right) K^* = \frac{\zeta}{\eta} Z^*. \qquad \text{(D.1)}$$

Substituting Q^* into (D.1) yields (18c). Equation (18c) shows that $Z^*/K^* > 0$ holds if and only if (18e) holds. Substituting $\dot{Q}_t^* = 0$ and Q^* into (16) yields R^*. From (11) and the Inada condition, K^* is uniquely determined by $R^* = \frac{\partial F(K^*,1)}{\partial K}$. □

E Proof of Proposition 3

Since inequality (18e) implies (18b), both steady states exist.

From $\frac{\partial F}{\partial K} = R$ and $\frac{\partial^2 F}{\partial K^2} < 0$, the following relationship holds.

$$\begin{aligned} K^{**} < K^* &\iff R^* < R^{**} \\ &\iff [(1-\lambda)r + \delta]\left(1 + \frac{\eta}{\zeta}r\right) < \frac{\eta}{\zeta}\frac{r\delta}{\lambda} + \delta \\ &\iff \text{(18e)}. \end{aligned}$$

Further, we obtain

$$\begin{aligned} K^{**} < K^* &\iff I^{**} < I^* \\ &\iff \frac{\zeta}{\eta}\lambda Q^{**}K^{**} < \frac{\zeta}{\eta}(\lambda Q^* K^* + Z^*) \\ &\iff v^{**}(\lambda K^{**}) < v^*(\lambda K^*) \end{aligned}$$

The first line uses $I = \delta K$. The second line uses (14). The last line uses (12). Thus, $K^{**} < K^*$ and $v^{**}(\lambda K^{**}) < v^*(\lambda K^*)$ hold if (18e) holds.

Assume that $F(K_t, N_t) = K^\alpha N^{1-\alpha}$ ($\alpha \in (0,1)$). From (18a) and (18c) , we have

$$\begin{aligned} Q^* K^* &= \left[\frac{\alpha Q^{*-\alpha}}{(1-\lambda)r + \delta}\right]^{\frac{1}{1-\alpha}} > 0, \\ Q^{**}K^{**} &= \left[\frac{\alpha\left(\frac{\eta\delta}{\zeta}\right)^{1-\alpha}\lambda^\alpha}{\frac{\eta\delta}{\zeta}r + \delta\lambda}\right]^{\frac{1}{1-\alpha}} \to 0 \quad \text{as} \quad \lambda \to 0. \end{aligned}$$

On the one hand, Q^*K^* is strictly positive. On the other hand, if $\lambda > 0$ is close to zero, $Q^{**}K^{**}$ is close to zero. Thus, if $\lambda > 0$ is not too large, we have $v^*(K^*) = Q^*K^* + Z^* > Q^{**}K^{**} = v^{**}(K^{**})$. □

F Proof of Proposition 4

Since R_t, K_t, and I_t are constant at the steady state, we have

$$\int_t^\infty (R_v K_v - I_v)\, e^{-r(v-t)} dv = \frac{1}{r}(RK - I), \tag{F.1}$$

where $(R, K, I) = (R^*, K^*, I^*)$ or $(R, K, I) = (R^{**}, K^{**}, I^{**})$. We first consider the steady state with $Z_t = 0$. At the steady state, $I^* = \delta K^*$ holds. Then, (F.1) can be rewritten as

$$\begin{aligned}\int_t^\infty (R_v K_v - I_v)\, e^{-r(v-t)} dv &= \frac{1}{r}\left(R^{**} - \delta\right) K^{**} \\ &= \frac{1}{r}\left(rQ^{**} + \delta - \delta\right) K^{**} \\ &= Q^{**} K^{**}.\end{aligned}$$

The second equality uses the second equation in (18a).

Next consider the steady state with $Z_t > 0$. From (16) and $\dot{Q}_t = 0$, we have $R^* K^* = (r+\delta) Q^* K^* - \frac{\zeta}{\eta}\lambda(Q^* - 1) Q^* K^*$. At the steady state, $I^* = \delta K^*$ holds. Thus, (F.1) can be rewritten as

$$\begin{aligned}\int_t^\infty (R_v K_v - I_v)\, e^{-r(v-t)} dv &= \frac{1}{r}\left[(r+\delta) Q^* K^* - \frac{\zeta}{\eta}\lambda(Q^* - 1) Q^* K^* - \delta K^*\right] \\ &= \frac{1}{r}\left[rQ^* K^* + (Q^* - 1)\left(\delta - \frac{\zeta}{\eta}\lambda Q^*\right) K^*\right] \\ &= \frac{1}{r}\left[rQ^* K^* + (Q^* - 1)\frac{\zeta}{\eta} Z^*\right] \\ &= Q^* K^* + Z^*.\end{aligned}$$

The third line uses (18c) and $Q^* = 1 + \frac{\eta}{\zeta} r$ in (18d). The last line again uses $Q^* = 1 + \frac{\eta}{\zeta} r$. $\square$

G Derivation of the Equilibrium dynamics

First, we derive the Bellman equation in the economy with $Z_t > 0$. Given $Z_t > 0$, Z_{t+dt} changes to zero with the probability εdt, meaning the the stock value of firm with capital K_t changes with εdt. Here, we drop index j from K_t since we focus on an equilibrium where $K_t^j = K_t$ for all j. The values of a defaulting and non-defaulting firm with K_t ((5) and (6)) are modified as follows:

$$\begin{aligned}v^D(K_t, t) = &(Y_t - w_t N_t - I_t + L_t) dt \\ &+ \frac{1 - \zeta dt}{1 + r dt}\left\{(1 - \varepsilon dt) v(K_t + dK_t, t + dt) + \varepsilon dt \cdot \tilde{v}(K_t + dK_t, t + dt)\right\}\end{aligned}$$

$$
\begin{aligned}
&+\frac{\zeta dt}{1+rdt}\left\{(1-\varepsilon dt)\left[v(K_t+dK_t,t+dt)-v\left(\lambda(K_t+dK_t),t+dt\right)\right]\right.\\
&\left.+\varepsilon dt\left[\tilde{v}(K_t+dK_t,t+dt)-\tilde{v}\left(\lambda(K_t+dK_t),t+dt\right)\right]\right\},\\
v^N(K_t,t)=&(Y_t-w_tN_t-I_t)dt\\
&+\frac{1}{1+rdt}\left\{(1-\varepsilon dt)v(K_t+dK_t,t+dt)+\varepsilon dt\cdot\tilde{v}(K_t+dK,t+dt)\right\},
\end{aligned}
$$

where $\tilde{v}(K_t,t)$ is the value of the firm with K_t when the sunspot shock is realized. The incentive-compatibility constraint for the firm with K_t is given by $v^N(K_t,t)\geq v^D(K_t,t)$ and the credit constraint (7) still holds.

With the sunspot shock, the maximization problem for the firm with K_t in the economy with $Z_t>0$ is modified as follows:

$$
\begin{aligned}
rv(K_t,t)=&\max_{N_t,I_t}(Y_t-w_tN_t-I_t)+\frac{\partial v\left(K_t,t\right)}{\partial K}(I_t-\delta K_t)+\frac{\partial v\left(K_t,t\right)}{\partial t}\\
&-\varepsilon\left[v(K_t,t)-\tilde{v}(K_t,t)\right],\ \text{s.t.}\quad(7).
\end{aligned}
$$

Except for the last term $-\varepsilon[v(K_t,t)-\tilde{v}(K_t,t)]$, the above maximization problem is same as (8). The first-order conditions (9) and (10) still hold. Once the sunspot shock is realized with the probability εdt, the value of the firm with K_t switches from $v(K_t,t)$ to $\tilde{v}(K_t,t)$, and thus the last term $\varepsilon[v(K_t,t)-\tilde{v}(K_t,t)]$ is interpreted as the capital gain (or loss) when the shock is realized.

We guess $v(K,t)=Q_tK+Z_t$ as in (12). We have $\tilde{v}(K,t)=\tilde{Q}_tK$. When (7) binds, aggregate investment is given by $I_t=\zeta(\lambda Q_tK_t+Z_t)/\eta$. Z_t and Q_t staisfy the following differential equations:[22]

$$
\dot{Z}_t=\left\{r+\varepsilon-\frac{\zeta}{\eta}(Q_t-1)\right\}Z_t, \tag{G.1}
$$

$$
\dot{Q}_t=(r+\varepsilon+\delta)Q_t-R_t-\frac{\zeta}{\eta}\lambda(Q_t-1)Q_t-\varepsilon\tilde{Q}_t, \tag{G.2}
$$

where $R_t=\partial F(K_t,1)/\partial K$ and $\tilde{Q}_t=J(K_t)$. With the realization of the shock, $Z_t>0$ immediately drops to zero. And Q_t jumps to $\tilde{Q}_t$, where $\tilde{Q}_t$ is on the saddle path converging

[22]The derivation procedure is the same as (15) and (16) in Proposition 1.

to the steady state with $Z^{**} = 0$. Later, we see how $\tilde{Q}_t$ is determined later.

G.1 Proof of Proposition 6

We denote the value of a variable X at the sunspot steady state by X^*_ε. Then, at the sunspot steady state, we have

$$\frac{Z^*_\varepsilon}{K^*_\varepsilon} = \frac{\eta}{\zeta}\delta - \lambda Q^*_\varepsilon > 0, \tag{G.3}$$

$$Q^*_\varepsilon = 1 + \frac{\eta}{\zeta}(r+\varepsilon), \tag{G.4}$$

$$R^*_\varepsilon = [(1-\lambda)r + \delta - \lambda\varepsilon]\, Q^*_\varepsilon + \varepsilon\left(Q^*_\varepsilon - \tilde{Q}\right), \tag{G.5}$$

$$\frac{\partial F(K^*_\varepsilon, 1)}{\partial K} = R^*_\varepsilon.$$

We first derive these for equations. Setting $\dot{Z}_t = 0$ in (G.1) yields (G.4) which uniquely determines Q^*_ε. Equation (G.4) shows that $Q^*_\varepsilon > 1$ holds and the credit constraint (7) binds, and thus we have $\eta I = \zeta(\lambda QK + Z)$. This equation and $I = \delta K$ yield (G.3). Equations (G.3) and (G.4) show that $Z^*_\varepsilon / K^*_\varepsilon > 0$ holds if and only if (22). Substituting (G.4) and $\dot{K}_t = 0$ into (G.2) yields (G.5).

Next, we derive $\tilde{Q}$ in the following two steps. In the first step, we show that $\tilde{Q}$ depends positively on R^*_ε by using the phase diagram after the sunspot shock (Figure A2).[23] Point b is the steady-state equilibrium with $Z^{**} = 0$ characterized by (18a).[24] Figure A2 shows that Point b is saddle-point stable and hence there is a unique saddle path converging to it. At the initial sunspot steady state, capital stock is K^*_ε and the marginal product of capital is equal to R^*_ε. At the moment of the sunspot shock, K^*_ε and R^*_ε do not change. Instead, Q_t jumps to $\tilde{Q}$, where $\tilde{Q}$ is on the unique saddle path (see Point a in Figure A2). As R^*_ε increases, Point a moves toward Point b. We denote this positive relationship between $\tilde{Q}$ and R^*_ε as $\tilde{Q} = \tilde{J}(R^*_\varepsilon)$.

In the next step, we substitute (G.4) and $\tilde{Q} = \tilde{J}(R^*_\varepsilon)$ into (G.5). Then, we have

$$R^*_\varepsilon = [(1-\lambda)(r+\varepsilon) + \delta]\left[1 + \frac{\eta}{\zeta}(r+\varepsilon)\right] - \varepsilon \tilde{J}(R^*_\varepsilon). \tag{G.6}$$

[23]See G.2 for the derivation of the phase diagram

[24]Since (22) implies implies (18b), the steady-state equilibrium with $Z^{**} = 0$ always exists if (22) holds.

The LHS increases with R^*_ε, whereas the RHS decreases with R^*_ε (see Figure A3). The intersection of the both sides uniquely determine R^*_ε and $\tilde{Q}$

G.2 Phase Diagram

We draw the phase diagram after the sunspot shock is realized (Figure A2). Suppose that $Z_t = 0$. When the credit constraints bind, K_t and Q_t satisfy

$$\dot{K}_t = \left(\frac{\zeta}{\eta}\lambda Q_t - \delta\right) K_t, \tag{G.7}$$

$$\dot{Q}_t = (r+\delta)Q_t - R_t - \frac{\zeta}{\eta}\lambda(Q_t - 1)Q_t. \tag{G.8}$$

From (G.7), $\dot{K}_t = 0$ locus is given by $Q_t = \frac{\eta\delta}{\zeta\lambda}$. From $R_t = \partial F(K_t, 1)/\partial K_t$, we have $dR_t/dt = F_{KK} dK_t/dt$ where $F_{KK} \equiv \frac{\partial^2 F}{\partial K^2} < 0$. Then, $\dot{R}_t = 0$ locus is also given by $Q_t = \frac{\eta\delta}{\zeta\lambda}$. Moreover, $\dot{R}_t > 0 \iff \dot{K}_t < 0$ holds because of $F_{KK} < 0$. Then, in the region below (above) $\dot{R}_t = 0$ locus, we have $\dot{R}_t > 0$ ($\dot{R}_t < 0$). See Figure A2.

From (G.8), $\dot{Q}_t = 0$ locus is given by $R_t = (r+\delta)Q_t - \frac{\zeta}{\eta}\lambda(Q_t - 1)Q_t \equiv \Gamma(Q_t)$. The function Γ has the following properties:

$$\Gamma(0) = 0 \ \text{ and } \ \Gamma\left(1 + \frac{r+\delta}{\frac{\zeta}{\eta}\lambda}\right) = 0,$$

$$\Gamma'(Q_t) = -2\frac{\zeta}{\eta}\lambda Q_t + (r + \delta + \frac{\zeta}{\eta}\lambda),$$

$$\Gamma'(0) = r + \delta + \frac{\zeta}{\eta}\lambda > 0,$$

$$\Gamma'(Q_t) = 0 \iff Q_t = \frac{r + \delta + \frac{\zeta}{\eta}\lambda}{2\frac{\zeta}{\eta}\lambda}.$$

In the region below (above) $\dot{Q} = 0$ locus, we have $\dot{Q}_t > 0$ ($\dot{Q}_t < 0$).

The phase diagram shows that the steady-state equilibrium with $Z^{**} = 0$ (point b) is saddle point stable.

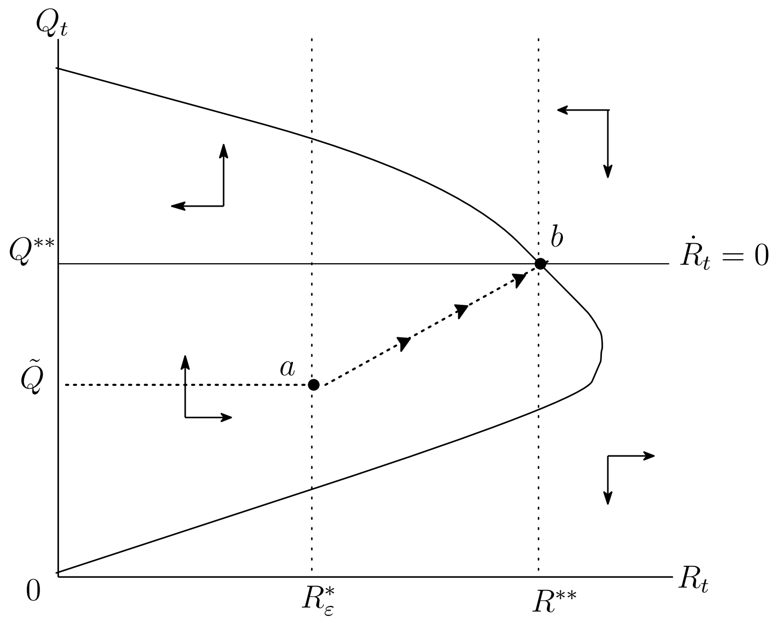

Figure A2 Phase Diagram

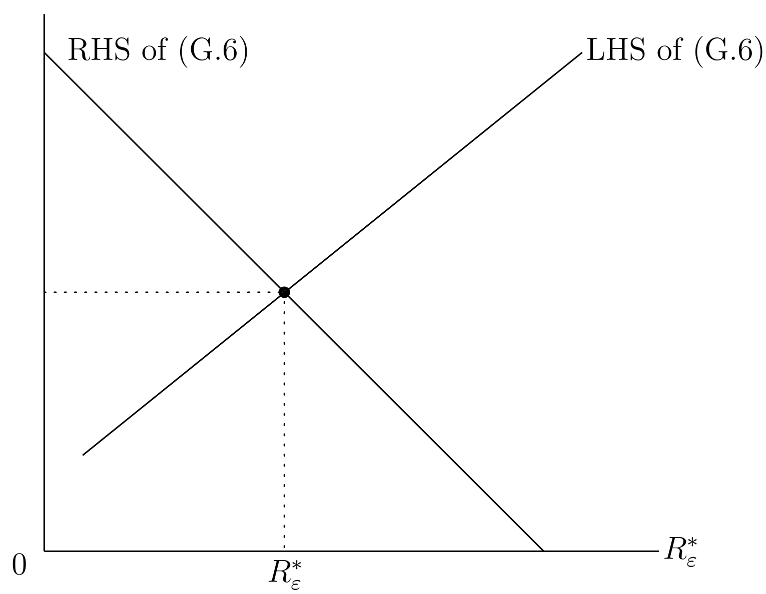

Figure A3 Existence of $\tilde{Q}$ and R_ε^*

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