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Asset Bubbles and Credit Constraints: Comment

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Asset Bubbles and Credit Constraints: Comment*

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Abstract

Miao and Wang (2018) propose a model in which a credit constraint produces multiple steady states with high and low stock prices. They interpret the high stock price as a stock price bubble. However, they do not provide a formal mathematical proof for the existence of bubbles, perhaps because their model is rather involved. We present a simple model that yields the same results as Miao and Wang (2018) and then show mathematically that bubbles do not exist.

Keywords: asset bubbles, fundamental value, credit constraints, self-fulfilling expectation, multiple equilibria

JEL classification numbers: E22, E44, G12

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1 Introduction

Miao and Wang (2018) present a continuous-time model with credit constraints and show that the interaction between stock prices and the credit constraints gives rise to multiple steady states with high and low stock prices. They interpret the high stock price as a bubble, though they do not provide a formal mathematical proof. This paper seeks to mathematically assess the presence of asset price bubbles in their model.

An asset bubble is defined as the difference between the market price of an asset and its fundamental value.[1] In the introduction of their paper, Miao and Wang (2018) discuss the existence of asset bubbles in the standard asset pricing theory under risk neutrality. Consider an asset that provides dividend D_t and trades at price p_t at time t. The asset satisfies the following no-arbitrage condition:

$$r_t = \frac{D_t + \dot{p}_t}{p_t},$$

where r_t is the interest rate of the economy. Solving this equation yields

$$p_0 = \lim_{T\to\infty} \left\{ \int_0^T D_t e^{-\int_0^t r_s ds} dt + p_T e^{-\int_0^T r_s ds} \right\},$$

where the present value of the dividend sequence, $\lim_{T\to\infty} \int_0^T D_t e^{-\int_0^t r_s ds} dt$, is the fundamental value of the asset. Thus, the asset price p_0 is equal to its fundamental value and consequently asset price bubbles do not exist if and only if

$$\lim_{T\to\infty} p_T e^{-\int_0^T r_s ds} = 0. \tag{1}$$

Despite their claim that asset bubbles exist in their model, Miao and Wang (2018) do not mathematically examine whether (1) is satisfied or not. In their model, the presence of idiosyncratic shocks and other auxiliary assumptions make it difficult to evaluate firm stock values (see the Appendix of this paper). Hence, this study first constructs a simple model that is bereft of those problems. Importantly, our model qualitatively provides the same results as Miao and Wang (2018). Subsequently, we derive an asset price equation and show mathematically that (1) holds. The fundamental value entirely determines the asset price. Asset bubbles cannot exist in their model and ours.

Some studies have examined qualitative as well as quantitative impacts of stock price bubbles, using the model proposed by Miao and Wang (2018). Examples of such studies include Miao and Wang (2015) and Miao, Wang, and Xu (2015). None of these studies provide formal mathematical proofs. Our results suggest that these applications might yield inaccurate implications regarding asset bubbles.

However, our results do not undermine the importance of the contribution of Miao and Wang (2018). By showing that asset prices could fluctuate considerably even without asset bubbles and could have significant impacts on the real economy, their model provides important insights about severe stagnation. Moreover, our simplified model significantly increases

[1] See Tirole (1985), Santos and Woodford (1997), Ljungqvist and Sargent (2018, Chapter 13), and Miao (2020, Section 13.6).

the applicability and usefulness of their model. Nevertheless, researchers must be careful not to apply the models to asset bubbles.

The remainder of the paper is structured as follows. Section 2 presents our model with credit constraints and reproduces the results in Miao and Wang (2018). Section 3 shows that (1) holds and hence asset bubbles cannot arise in our model and that of Miao and Wang (2018). In Section 4, we explain that self-fulfilling expectation leads to multiple equilibria, where stock prices can be low or high. Thus, both Miao and Wang (2018) and our model are categorized to the class of models of self-fulfilling expectations, proposed by Azariadis (1981) and Farmer (1999). Section 5 reviews the non-existence conditions of bubbles in the literature and show that these conditions are satisfied in Miao and Wang (2018) and our model. The conclusion is presented in Section 6.

2 A Simplified Model of Miao and Wang (2018)

There is a continuum of firms whose measure is one. Each firm is indexed by $j \in [0,1]$. The stock price of firm j is V_t^j and its dividend is D_t^j. There is no uncertainty in the economy.

2.1 Households

As in Miao and Wang (2018), the representative household is endowed with a linear utility, $U_t = \int_t^\infty c_s e^{-r(s-t)} ds$, where $r > 0$ is a (constant) subjective discount rate and c_t is consumption at time t. The representative household inelastically supplies one unit of labor. Its budget constraint is $\int V_t^j \dot{\phi}_t^j dj + c_t = w_t + \int D_t^j \phi_t^j dj$, where w_t is the wage rate, ϕ_t^j denotes the holdings of firm j's share, and thus $\dot{\phi}_t^j$ denotes the purchasing of firm j's share. As in Miao and Wang (2018), utility maximization yields

$$rV_t^j = D_t^j + \dot{V}_t^j, \tag{2}$$

and the transversality condition is

$$\lim_{T\to\infty} V_T^j \phi_T^j e^{-rT} = \lim_{T\to\infty} V_T^j e^{-rT} = 0. \tag{3}$$

In (3), we use the stock market clearing condition $\phi_t^j = 1$.

2.2 Firms

Each firm produces a single final good that can be used for consumption and capital accumulation. We describe firm j's behavior in the infinitesimally short time interval between time t and $t + dt$. Here, t and dt can differ among firms because different firms could have different planning periods.[2] At time t, firm j owns the capital stock K_t^j. In the time interval $[t, t + dt]$, firm j employs $N_t^j dt$ units of labor. At the end of this interval (time $t + dt$), firm j completes its final good production:

$$y_t^j dt = (K_t^j)^\alpha (N_t^j)^{1-\alpha} dt, \quad \alpha \in (0,1).$$

[2] Putting firm index j on the time interval, as in $[t^j, t^j + dt^j]$, might be more precise. However, to make the notation simple, we omit index j from the time interval.

We normalize the price of the final good to one. At the end of the interval (time $t+dt$), firm j sells the final good to households as a consumption good and other firms as an investment good, earning the revenue $y_t^j dt$.

At the same time, firm j invests $I_t^j dt$ units of the final good produced by other firms in the capital stock during the interval of $[t, t+dt]$. We assume that firm j's own output cannot be used for firm j's investment. Then, firm j's capital stock accumulates according to

$$dK_t^j = (I_t^j - \delta K_t^j)dt, \tag{4}$$

where $\delta > 0$ is the capital depreciation rate.

Note that there is a timing mismatch. Firm j invests in capital at the start of the time interval (time t), while it earns revenue from production at the end of the time interval (time $t+dt$). To finance its capital investment, firm j makes a credit sales contract with other firms. More precisely, firm j procures $I_t^j dt$ units of the final good at time t with a commitment to repay it at time $t+dt$ after it earns revenue.[3] No interest is charged on this payment. Assume that $\eta \in [0,1]$ fraction of capital investment must be financed by the credit sales contract. Firm j must satisfy the following constraint:

$$\eta I_t^j dt \leq L_t^j dt, \tag{5}$$

where $L_t^j dt$ denotes the fund available to firm j through the credit sales contract. As in Miao and Wang (2018), the wage payment $w_t N_t^j$ is not subject to timing mismatch.

Firm j has the option of default. If firm j defaults, it loses $\lambda K_t^j dt$ units of capital as a defaulting cost. The defaulting firm is detected with probability ζdt, whereby its revenue is confiscated, and it faces a permanent exclusion from the market. The defaulting firm is not detected with probability $1 - \zeta dt$ and does not repay $L_t^j dt$. Then, firm continues its operation.

Incentive Constraint: Let $v^N(K_t^j, t)$ and $v^D(K_t^j, t)$ be the stock values of a non-defaulting and defaulting firm with capital stock K_t^j at time t, respectively. We assume that $v^N(K_t^j, t)$ and $v^D(K_t^j, t)$ are differentiable with respect to both arguments. Let us define

$$v(K_t^j, t) = \max\left\{v^N(K_t^j, t), v^D(K_t^j, t)\right\}.$$

A defaulting firm does not repay $L_t^j dt$ and escapes with probability $1 - \zeta dt$, incurring the defaulting cost of $\lambda K_t^j dt$ units of capital. Hence, the value of a defaulting firm satisfies

$$v^D(K_t^j, t) = (y_t^j - w_t N_t^j - I_t^j + L_t^j)dt + \frac{1 - \zeta dt}{1 + rdt} v\left((1 - \lambda dt)K_t^j + dK_t^j, t + dt\right).$$

As a non-defaulting firm repays $L_t^j dt$, its value satisfies

$$v^N(K_t^j, t) = (y_t^j - w_t N_t^j - I_t^j)dt + \frac{1}{1 + rdt} v\left(K_t^j + dK_t^j, t + dt\right). \tag{6}$$

[3] At time $t + dt$, firm j repays by using sales of consumption goods to households, the $1 - \eta$ fraction of investment goods sales to other firms, and repayments from other firms.

If $v^N(K_t^j,t) \geq v^D(K_t^j,t)$, firms have no incentive to default. After some simple calculations,[4] we can rewrite this inequality as

$$\eta I_t^j \leq \zeta v\left(K_t^j,t\right) + \frac{\partial v\left(K_t^j,t\right)}{\partial K}\lambda K_t^j. \tag{7}$$

As the above inequality puts a constraint on capital investment financed by credit, we simply call (7) the *credit constraint.*

Optimization and First-order Conditions: As long as (7) is satisfied, we have $v(K_t^j,t) = v^N(K_t^j,t)$. From (6), the maximization problem of firms is formulated as follows:

$$rv(K_t^j,t) = \max_{N_t^j,I_t^j}(y_t^j - w_tN_t^j - I_t^j) + \frac{v\left(K_t^j + dK_t^j, t+dt\right) - v\left(K_t^j,t\right)}{dt}, \quad \text{s.t. } (4) \text{ and } (7).$$

If we take the limit $dt \to 0$ in the above equation, we have

$$rv(K_t^j,t) = \max_{N_t^j,I_t^j}(y_t^j - w_tN_t^j - I_t^j) + \frac{\partial v\left(K_t^j,t\right)}{\partial K}(I_t^j - \delta K_t^j) + \frac{\partial v\left(K_t^j,t\right)}{dt}, \quad \text{s.t.} \quad (7). \tag{8}$$

The first-order conditions of the problem are given by

$$w_t = (1-\alpha)(K_t^j)^\alpha(N_t^j)^{-\alpha}, \tag{9}$$

$$\eta\mu_t = \frac{\partial v\left(K_t^j,t\right)}{\partial K} - 1, \tag{10}$$

where μ_t is the Lagrangian-multiplier associated with (7). Using (9), we obtain

$$y_t^j - w_tN_t^j = \alpha\left(\frac{1-\alpha}{w_t}\right)^{\frac{1-\alpha}{\alpha}} K_t^j \equiv R_tK_t^j. \tag{11}$$

In (10), $\partial v\left(K_t^j,t\right)/\partial K$ is Tobin's marginal q. If and only if the credit constraint (7) binds, Tobin's marginal q is greater than investment cost (=1).

Deriving the Key Differential Equations: As in Miao and Wang (2018), we conjecture that $v(K_t,t)$ is linear in K_t^j as follows:

$$v(K^j,t) = Q_tK^j + B_t. \tag{12}$$

Here, we intentionally drop time index t from K^j to emphasize that the functional form of $v(K^j,t)$ depends on time t, which is reflected by the time dependence of Q_t and B_t. Then, we have

$$\frac{\partial v(K^j,t)}{\partial t} = \dot{Q}_tK^j + \dot{B}_t \quad \text{and} \quad \frac{\partial v(K^j,t)}{\partial K} = Q_t. \tag{13}$$

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[4]Using the approximation of $dt \cdot dt = 0$, we can rewrite this inequality as

$$L_t^j \leq \zeta v\left((1-\lambda dt)K_t^j + dK_t^j, t+dt\right) + \frac{v\left(K_t^j + dK_t^j, t+dt\right) - v\left(K_t^j + dK_t^j - \lambda K_t^j dt, t+dt\right)}{dt}.$$

If we take the limit $dt \to 0$ and use (5) in the above inequality, we obtain (7).

As the first equation shows how the functional form changes over time, we need not differentiate one of the arguments, K_t^j, with respect to t.

The following Proposition is analogous to Proposition 1 in Miao and Wang (2018).

Proposition 1 *Suppose that* $Q_t > 1$. *Then,* (7) *binds and we have*

$$I_t^j = \frac{\zeta + \lambda}{\eta} Q_t K_t^j + \frac{\zeta}{\eta} B_t. \tag{14}$$

B_t *and* Q_t *satisfy the following differential equations:*

$$\dot{B}_t = \left\{ r - \frac{\zeta}{\eta}(Q_t - 1) \right\} B_t, \tag{15}$$

$$\dot{Q}_t = (r + \delta)Q_t - R_t - \frac{\zeta + \lambda}{\eta}(Q_t - 1)Q_t, \tag{16}$$

as well as the transversality conditions

$$\lim_{T \to \infty} Q_T K_T^j e^{-rT} = \lim_{T \to \infty} B_T e^{-rT} = 0. \tag{17}$$

(Proof) Suppose that $Q_t > 1$. Then, (7) binds and we must have (14). By using (7), (11), (12), and (13), we rewrite (8) as

$$\begin{aligned} r(Q_t K_t^j + B_t) &= R_t K_t^j - I_t^j + Q_t(I_t^j - \delta K_t^j) + \dot{Q}_t K_t^j + \dot{B}_t \\ &= R_t K_t^j - \delta Q_t K_t^j + (Q_t - 1) I_t^j + \dot{Q}_t K_t^j + \dot{B}_t \\ &= R_t K_t^j - \delta Q_t K_t^j + \frac{Q_t - 1}{\eta} \left\{ (\zeta + \lambda) Q_t K_t^j + \zeta B_t \right\} + \dot{Q}_t K_t^j + \dot{B}_t. \end{aligned} \tag{18}$$

As the relationship (18) holds for any $K_t^j > 0$, then (15) and (16) must hold. Further as $V_t^j = v(K_t^j, t)$ holds in equilibrium, (3) ensures (17). □

Let us set $\frac{\zeta+\lambda}{\eta} \equiv \xi$ and $\frac{\zeta}{\eta} \equiv \pi (= 1)$ in (14), (15), and (16). Then, (14), (15), and (16) are exactly the same as equations (19), (20), and (21) in Miao and Wang (2018), respectively. The following two comments are noteworthy. First, in their model, π is the fraction of firms with an investment opportunity per unit of time. As all firms in our model always have an investment opportunity, $\pi = 1$ naturally holds. Second, in Miao and Wang (2018), a large ξ means that firms face a loose credit constraint. In our model, large ζ and λ mean that the default costs are large and hence firms have less incentive to default. A small η suggests that investment is less subject to the credit constraint. Thus, a large $\frac{\zeta+\lambda}{\eta}$ indicates a loose credit constraint in our model. Therefore, (14), (15), and (16) in our model can be interpreted in the same way as equations (19), (20), and (21) in Miao and Wang (2018), respectively. Thus, our model provides the same results as Miao and Wang (2018).

Miao and Wang (2018) show that under some conditions, two steady-state equilibria exist, one with $B_t = 0$ and the other with $B_t > 0$. They interpret a strictly positive $B_t (> 0)$ as a stock price bubble. However, partly due to the idiosyncratic shocks, it is quite difficult to verify if (1) holds or not (see the Appendix of this paper).

3 Asset Price and Bubbles

As our model does not include any idiosyncratic shocks and uncertainty, it is fairly easy to examine if (1) holds. For simplicity, let us focus on an equilibrium where $K_t^j = K_t$ for all j and omit firm index j.

In the utility maximization problem of the household, we denote the stock price of a firm as V_t. As $V_t = v(K_t, t)$ holds in equilibrium, we have

$$\dot{V}_t = \frac{\partial v(K_t,t)}{\partial K}\frac{dK_t}{dt} + \frac{\partial v(K_t,t)}{\partial t} = \frac{\partial v(K_t,t)}{\partial K}(I_t - \delta K_t) + \frac{\partial v(K_t,t)}{\partial t}.$$

Thus, the first line of (18) can be rewritten as

$$rV_t = R_t K_t - I_t + \dot{V}_t. \tag{19}$$

We have derived the above equation by solving the firm's optimization problem. This equation must be consistent with the no-arbitrage condition (2) that is derived from the households' optimization problem. Comparison between (2) and (19) yields

$$D_t = R_t K_t - I_t. \tag{20}$$

The dividend of a firm is equal to the operating profits minus investment and is distributed to households. By solving (19), we derive the familiar asset price equation:

$$V_t = \int_t^\infty (R_v K_v - I_v)\, e^{-r(v-t)} dv + \lim_{T\to\infty} V_T e^{-r(T-t)}. \tag{21}$$

The first term, the present value of the future dividend sequence $\lim_{T\to\infty}\int_0^T D_t e^{-\int_0^t r_s ds} dt$, is the fundamental value of the asset. The second term is the bubble term.

As mentioned earlier, Miao and Wang (2018) interpret a strictly positive $B_t (> 0)$ as a stock price bubble. However, Miao and Wang (2018) do not derive the asset price equation as in (21) and hence they do not check if the bubble term is zero or strictly positive. In the Appendix of this paper, we briefly mention several reasons why evaluating the bubble term is difficult in their model.

The following proposition provides one of the two main results:

Proposition 2 *The transversality condition* (3) *(or* (17)*) ensures that*

$$\lim_{T\to\infty} V_T e^{-r(T-t)} = 0. \tag{22}$$

Other than relying on the transversality condition (3) (or (17)), there are several ways to prove (22). For example, as the economy converges to one of the steady states where V_t is constant over time, (22) must hold.

Another important proof is to directly evaluate the fundamental value and measure the deviation from the asset value. We do this by focusing on the steady state with $B_t > 0$ and $\dot{B}_t = 0$. Variables with an asterisk, such as Q^*, K^*, and I^*, stand for steady state values.

Proposition 3 *At the steady state with $B_t > 0$, the fundamental value of the asset is given by*

$$\int_t^\infty (R_v K_v - I_v)\, e^{-r(v-t)} dv = Q^* K^* + B^*. \tag{23}$$

(Proof) Since R_t, K_t, and I_t are constant at the steady state, we have

$$\int_t^\infty (R_v K_v - I_v)\, e^{-r(v-t)} dv = \frac{1}{r}\left(R^* K^* - I^*\right). \tag{24}$$

From (15), we have $Q^* = 1 + \eta r/\zeta$. From (16), Q^*, and $\dot{Q}_t = 0$, we have $R^* = Q^*\delta - \lambda(Q^* - 1)Q^*/\eta$. If we multiply both sides of this equation by K^* and then use $0 = \dot{K}_t = I^* - \delta K^*$, we obtain $R^* K^* = Q^* \delta K^* - \lambda(Q^* - 1)Q^* K^*/\eta = Q^* I^* - \lambda(Q^* - 1)Q^* K^*/\eta$. Thus, (24) can be rewritten as

$$\begin{aligned} \int_t^\infty (R_v K_v - I_v)\, e^{-r(v-t)} dv &= \frac{1}{r}\left\{Q^* I^* - \frac{\lambda}{\eta}(Q^* - 1)Q^* K^* - I^*\right\} \\ &= \frac{1}{r}(Q^* - 1)\left(I^* - \frac{\lambda}{\eta} Q^* K^*\right). \end{aligned} \tag{25}$$

Substituting (14) into (25), we have

$$\int_t^\infty (R_v K_v - I_v)\, e^{-r(v-t)} dv = \frac{1}{r}(Q^* - 1)\frac{\zeta}{\eta}\left(Q^* K^* + B^*\right).$$

Since $Q^* = 1 + \eta r/\zeta$, we finally obtain (23). □

Clearly, (23) implies (22). Propositions 2 and 3 demonstrate the following crucial points:

1. Condition (1) holds. Thus, the asset price bubble does not exist.
2. The asset price is entirely determined by the fundamental value.
3. The term B_t is a part of the fundamentals.

4 Interpretation of Multiple Equilibria

We know that the stock price is determined by the fundamental value. Thus, the occurrence of multiple equilibria is not attributable to stock price bubbles. To understand the mechanism behind the multiple equilibria and the high stock price, we replicate the credit constraint of firm j, (7), assuming that it is binding:

$$\eta I_t^j = \zeta V_t^j + \frac{\partial V_t^j}{\partial K} \lambda K_t^j.$$

As the asset price is entirely determined by the fundamental value, it depends on the (expected) future dividend stream

$$V_t^j = \int_t^\infty D_v^j e^{-r(v-t)} dv, \qquad D_t^j = R_t K_t^j - I_t^j.$$

Suppose that the future dividend stream is expected to increase. This expectation immediately increases the asset price V_t^j. As a result, the credit constraint of firm j is eased, allowing firm j to expand investments and accumulate a greater amount of capital. Consequently, the future dividend stream indeed increases. The initial optimistic expectation about the dividend stream becomes self-fulfilling.

This self-fulfilling expectation leads to the existence of multiple equilibria, where stock prices can be high or low. Thus, both Miao and Wang (2018) and our model belong to the class of multiple-equilibria models in which self-fulfilling beliefs influence equilibrium outcomes, proposed by Azariadis (1981) and Farmer (1999).

5 Non-existence of Bubbles in the Literature

This section reviews the (non-)existence conditions of bubbles in the existing literature.

First, rational bubble models usually assume some form of heterogeneity to ensure that asset trades take place among economic agents. If the time interval $[t, t+dt]$ is common for all firms, no heterogeneity is present in our model. Nevertheless, term B_t exists, which suggests that it might not be the bubble term.

Second, many authors repeatedly show that asset bubbles exist if and only if the economic growth rate exceeds the interest rate in the economy without bubbles. In Miao and Wang (2018) and our model, the economy is not growing while the interest (discount) rate is strictly positive. Our result of no existence of asset bubbles is in line with the literature.

The third point is related to the second one. Kocherlakota (1992) and Santos and Woodford (1997) present the (non-)existence conditions of asset bubbles in exchange economies. Proposition 4 in Kocherlakota (1992) shows that in any equilibrium with asset bubbles, the discounted value of the aggregate endowment is infinite.[5] Santos and Woodford (1997) prove that asset bubbles do not exist if the net supply of the asset is positive and the discounted value of the aggregate endowment is finite. This result may not directly apply to production economies. However, as the economy approaches a steady state, the discounted value of the aggregate output is finite in both Miao and Wang (2018) and our model. Our non-existence result of bubbles is consistent with these two influential studies.

Fourth, Hirano and Toda (2023) show that in a discrete-time setting there is an asset price bubble if and only if $\sum_{t=1}^{\infty} D_t/V_t < \infty$, where D_t and V_t are the dividend and the asset price in period t, respectively. In Miao and Wang (2018) and our model, because the economy converges to a steady state in which both the dividend and asset price are constant over time, the sum of D_t/V_t is infinite. Hirano and Toda (2023)'s existence condition of bubbles is violated.

Finally, according to Kamihigashi (1998), the asset price becomes equal to the fundamental value under a linear utility in a Lucas tree economy.[6] His result might not apply directly to our production economy. However, Miao and Wang (2018) and our model satisfy his conditions for the non-existence of bubbles.

[5] Kocherlakota and Toda (2023) offer a revised proof for this proposition.

[6] Linear utility functions satisfy conditions (L1), (L2), and (U2) in Theorem 5 of Kamihigashi (1998).

6 Conclusion

We construct a simple model that is qualitatively the same as the model of Miao and Wang (2018). Based on Condition (1), asset bubbles are absent in both their model and ours. The non-existence of bubbles in Miao and Wang (2018) and our model is consistent with the literature. Thus, applying their model to asset bubbles leads to erroneous predictions.

Appendix

In the model of Miao and Wang (2018), checking if (1) holds is extremely difficult; for several reasons, it is not clear how the no-arbitrage condition derived from the households' optimization problem is associated with that derived from the firms' optimization. We list three reasons below.

1. When Miao and Wang (2018) formulate the firm's maximization problem in equation (9) of their paper, they multiply the dividend of non-investing firms D_{0t}^j by Δ while they do not multiply the dividend of investing firms D_{1t}^j by Δ. In the no-arbitrage condition derived from the households' maximization problem, the expected dividend D_t^j is multiplied by Δ (see equation (4) of their paper). The relationship among D_t^j, D_{0t}^j, and D_{1t}^j is unclear, which makes it difficult to evaluate the fundamental values. Problematically, if D_{1t}^j is multiplied by Δ, it will disappear from equation (9).

2. The timing assumption regarding the investment opportunity is confusing. The arrival of investment opportunities follows a Poisson process with a rate of π. Thus, each firm has an investment opportunity between t and $t + \Delta$ with probability $\pi\Delta$.

   However, Miao and Wang (2018) often write that a firm has an investment opportunity at the initial point of time interval $[t, t+\Delta]$ or at time t (see the last paragraph of p.2598, the paragraph that includes equation (13) on p.2600, and the first line of p.2603, for example). This claim is problematic for the following reasons.

   (a) In a continuous-time setting, the probability with which an investment opportunity hits a firm at time t is zero.

   (b) If a firm has an investment opportunity at time t as Miao and Wang (2018) write, there is no uncertainty about the investment opportunity in period $(t, t+\Delta]$, which contradicts equation (9) of their paper.

3. In the firms' optimization problem, it is unclear if capital K_t^j is a stock variable or not. On the one hand, it depreciates at a positive rate δ. On the other hand, the last term in equation (14) of their paper implies that the capital of firm j could jump discontinuously from K_t^j to $K_{1,t}^j$ at a rate of π. If K_t^j is a stock variable, such a jump is impossible.

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