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On the observable restrictions of limited consideration models: theory and application

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On the observable restrictions of limited consideration models: theory and application

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Abstract

This paper develops revealed preference analysis for limited consideration models. A revealed preference test is given for the decision model obeying two well-established hypotheses on a decision maker's consideration: the *attention filter* property and *competition filter* property. We also provide a test for the two-step decision model called the *(transitive) rational shortlist method*. As an application, we conducted a simulation to compare the relative strength of observable restrictions across leading models, in addition to an experiment to compare models in terms of Selten's index, which is a measure for plausibility of a model in explaining a given data set.

KEYWORDS: Revealed preference; Limited consideration; Limited attention; Rational shortlisting; Bronars' test; Selten's index

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1 Introduction

Motivated by well-established evidence that a decision maker (DM) does not consider all feasible alternatives, various decision models under limited consideration have been proposed. There, some feasible alternatives are a priori excluded from an agent's consideration due to the limitation of recognition capacity and/or due to the shortlisting according to some criteria different from her preference (e.g., psychological restrictions, a preference on categories rather than alternatives, and others). Each of those models is conceptually insightful and plausible, and they have different interpretations and structures on a DM's consideration, which could in turn result in different empirical implications.

Given this, it is important to have a tool for testing each decision model based on observed choice behavior of a DM. Following the pioneering work by De Clippel and Rozen (2018a), this paper puts forward an empirical revealed preference theory of limited consideration models. As a theoretical contribution, we establish revealed preference tests for several important decision models that are not covered in the literature. Similar to De Clippel and Rozen's setting, our tests are based on a data set in the form of $\mathcal{O} = \{(a^t, A^t)\}_{t\in\mathcal{T}}$, where $\mathcal{T}$ is the set of indices of observations, and for each $t \in \mathcal{T}$, a^t is the chosen alternative from the set of feasible alternatives A^t. In particular, it suffices to observe choices on *some* feasible sets, rather than an entire choice function, and we do not require any information on a DM's consideration. Applying our revealed preference tests as well as existing ones, we provide the following two types of data analyses. First, by simulation we numerically compare the relative strength of observable restrictions across various models, and second, we carried out an experiment to compare the models in terms of Selten's index, which measures the plausibility of a model in explaining data sets.

In general, a limited consideration model can be described as a pair of preference and consideration mapping $(\succ, \Gamma)$, of which the latter specifies a consideration set $\Gamma(A)$ for each feasible set A. Throughout this paper, we stick to the case of a DM's preference $\succ$ being a strict preference, i.e., $\succ$ is complete, asymmetric and transitive.[1] The property of Γ varies across models, and it in turn specifies the nature of a model. Amongst others, we look at limited

[1]There are some models in which a DM's preference satisfies less properties (e.g., it only obeys asymmetry rather than transitivity), and revealed preference tests for these cases are treated in De Clippel and Rozen (2018b). On the other hand, as also referred to in Lleras, Masatlioglu, Nakajima and Ozbay (2017), the results from an eye-tracking experiment by Reutskaja, Nagel, Camerer and Rangel (2011) broadly supports the hypothesis of a strict preference.

consideration models with Γ having the following structures that are shared by many decision procedures. We say that a consideration mapping Γ is an *attention filter (AF)*, if removal of unrecognized alternatives does not affect the consideration set, namely, $x \in A$ and $x \notin \Gamma(A) \Longrightarrow \Gamma(A\backslash x) = \Gamma(A)$. On the other hand, Γ is referred to as a *competition filter (CF)*, if the set of unrecognized alternatives is monotonic with respect to the set inclusion order of feasible sets, i.e., $x \in A' \subset A''$ and $x \notin \Gamma(A') \Longrightarrow x \notin \Gamma(A'')$. When Γ has both AF and CF structures, then we refer to it as a *competitive attention filter (CAF)*. The axiomatic characterizations of a model with AF and that with CF/CAF are respectively given by Masatlioglu, Nakajima and Ozbay (2012) and Lleras, Masatlioglu, Nakajima and Ozbay (2015, 2017) based on a choice function of a DM.[2] On the other hand, De Clippel and Rozen (2018a) provides a revealed preference test for a model with AF based on limited data, and a similar test for a model with CF is essentially shown by Dean, Kibris and Masatlioglu (2017).[3] In this paper, as a follow-up of these papers, we establish a revealed preference test *a lá* De Clippel and Rozen for a model with CAF. Note that even if a data set passes both a test for AF and that for CF, it may not be consistent with any model with CAF.

There are a number of real-world examples that generate CAF. If for example, a DM pays attention to: (a) n-most advertised commodities; (b) all commodities of a specific brand, and if there are none available, then all commodities of another specific brand; or (c) n-top candidates in each field in job markets, then all of them derive CAF. More importantly, using classical results in social choice theory, a CAF can be characterized in the following two ways, both of which provide clear economic interpretations of testing a model with CAF:

(i) Suppose that a DM has multiple criteria $\{\mathcal{R}_i\}_{i \in I}$ that compare alternatives and that she only considers alternatives that are the best in terms of some $\mathcal{R}_i$. If every $\mathcal{R}_i$ is complete, asymmetric and transitive, then a DM's consideration mapping is a CAF, and conversely, every CAF can be represented as such. If each $\mathcal{R}_i$ is a general binary relation, then the above procedure is equal to the *rationalization* model in Cherepanov, Feddersen and Sandroni (2013), of which the consideration mapping is characterized by CF. Thus, testing a model with CAF can be interpreted as testing whether a DM's consideration is formed by using criteria with order structure, while testing CF is equivalent to testing

[2]Lleras et al. (2015) is a working paper version of Lleras et al. (2017).

[3]Dean et al. (2017) provides a revealed preference test for a status-quo bias model where a DM's consideration is a CF that depends on a status-quo. It can be easily adjusted to our context. Note also that the first version of De Clippel and Rozen (2018a) dates back to 2012, and, to the best of the authors knowledge, it is the first paper to look at revealed preference analysis of limited consideration models with limited data.

the rationalization model itself.

(ii) Suppose that a set of feasible alternatives A is represented as $A = A' \cup A''$. In fact, if (and only if) a DM's consideration mapping Γ is a CAF, then it holds that $\Gamma(A) = \Gamma(\Gamma(A') \cup A'') = \Gamma(A' \cup \Gamma(A''))$, which is referred to as *path independence (PI)*. Through an inductive argument, as its name suggests, PI ensures that the formation of a consideration set can be decomposed into an arbitrary path of the formations of consideration sets on smaller feasible sets. Put otherwise, when a DM sequentially narrows down alternatives like a tournament, the order of treatments does not affect the resulting consideration set. Thus, testing a model with CAF can be interpreted as testing this type of robustness to a sequential formation of a consideration set.

As an important special case of a limited consideration model with CF/CAF, we also provide revealed preference analysis on the following two-step decision procedure so called the *rational shortlisting method*. There, a DM makes a shortlist of alternatives that are undominated in terms of some acyclic binary relation, and then maximize her preference within this shortlist. The axiomatic characterization of this type of model is firstly given by Manzini and Mariotti (2007) and an important variation, where the first step preference is asymmetric and transitive, is studied by Au and Kawai (2011). For simplicity, we refer to a consideration mapping derived from these models respectively as a *rational shortlist (RS)* and a *transitive rational shortlist (TRS)*. In fact, a (T)RS is a CF (CAF) obeying the additional axiom called *expansion (EX)*: for every A' and A'', $\Gamma(A') \cap \Gamma(A'') \subset \Gamma(A' \cup A'')$. Thus, our revealed preference test for a model with (T)RS can be interpreted as a test for a model with CF (CAF) obeying EX.

Given tools for testing limited consideration models, we apply them to look at the empirical aspects of models. Specifically, gathering together with existing tests, now we can test models with AF, CF, CAF and (T)RS. It is obvious that limited consideration models are relatively permissive compared to the rational choice model, and there are several subclass/superclass relations within limited consideration models (e.g., CAF is obviously stronger than both of AF and CF). Meanwhile, it is not at all clear *how* relatively restrictive/permissive the models are. Following Bronars (1987), we generate random choices and apply our tests to see the fraction of data that are consistent with each model. Provided that observable restriction of each model depends on the structure of feasible sets, we repeat the above procedure over randomly generated profiles of feasible sets. In our simulation, we stick to the environment with 20 feasible sets, each of which contains 2 – 8 alternatives out of 10 alternatives. Our result shows that the strength of observable restriction is strikingly different across models. A model with AF is very hard to reject with average pass rate of random data exceeding 99%, and a model with CF is also permissive with average pass rate exceeding 60%. However, the joint of them, or a model with CAF, is far more restrictive with average pass rate being less than 4%. The rational shortlist models both have strong testing power: the average pass rate of RS is less than 3% and that of TRS is less than 0.1%. In other words, while AF itself is very hard to reject, imposing it on top of CF or RS could drastically strengthen observable restrictions (note that an RS that also obeys AF property is a TRS). A similar argument applies to imposing EX on top of CF or CAF, given that (T)RS is the conjunction of CF (CAF) and EX.

Furthermore, based on an experimental data set, we compare models in terms of *Selten's index*, which is a measure for evaluating a model as an explanation of data. Given choice data of subjects, Selten's index of a model is practically calculated as the difference between pass rate of the revealed preference test of actual data and that of randomly generated choices. Loosely speaking, a model is highly evaluated if (i) it can well explain observed choices, while (ii) its observable restriction is strong (see Selten (1991) and Beatty and Crawford (2011)). In our baseline experiment, we adopted one profile of feasible sets generated in the simulation part, i.e., each subject was asked to make choices on 20 feasible sets containing 2 – 8 alternatives out of 10 alternatives. Amongst 113 subjects, 33% of them were consistent with the rational choice model, about 60% were consistent with RS/TRS, and the pass rates for AF, CF and CAF exceeded 90% (nobody failed AF test). On the other hand, in terms of Selten's index, CAF distinctively performed well. As a comparative experiment, we also collected choice data with smaller feasible sets, where each subject was given 20 feasible sets containing 2 – 5 alternatives out of 10 alternatives. In this case, TRS achieved the highest value of Selten's index.

Organization of the paper: In Section 2, we briefly review limited consideration models dealt with in this paper, as well as some characterization results known in the literature. The theoretical heart of our paper lies in Section 3: we firstly provide a basic idea of our approach in testing limited consideration models in Section 3.1, and then provide revealed preference tests for a model with CAF and that with (T)RS respectively in Sections 3.2 and 3.3. In Section 4, we apply our tests to simulation and experimental data. The substantial parts of proofs of main theorems are given in Appendix I. From a technical perspective, our revealed preference tests have a similar mathematical structure, so called *acyclic satisfiability*, to the test for a model with AF in De Clippel and Rozen (2018a). As rigorously shown in their paper, this class of problems can be computationally challenging, and our tests also share that computational issue. Nevertheless, features of the tests allow us to employ a computing method called *backtracking*, which is an efficient search method in dealing with combinatorial problems.[4] We adopted this method in our testing algorithms, and actually applied them in the simulation and experiment. The basic idea of backtracking and our algorithm is given in Appendix II. The detail of experimental setting is summarized in Appendix III.

2 Limited consideration models

Consider a single-agent decision problem where X is a finite set of alternatives, and $\succ$ is a complete, asymmetric and transitive preference of a DM, to which we refer as a *strict preference*. If a DM obeys the *rational choice* model, then for every feasible set $A \in 2^X$, she maximizes her strict preference on A. On the other hand, in limited consideration models, either consciously or unconsciously, a DM makes a shortlist of alternatives, and then she maximizes her preference on that shortlist. That is, there exists a *consideration mapping* $\Gamma : 2^X \to 2^X$ such that $\Gamma(A) \subset A$ for every $A \in 2^X$, and a DM maximizes her strict preference on $\Gamma(A)$, rather than A itself. Given a consideration mapping Γ, $\Gamma(A)$ is referred to as a *consideration set* on A, and, in what follows, we always assume that $A \neq \varnothing$ and $\Gamma(A) \neq \varnothing$ for every A. We call a pair of strict preference and consideration mapping $(\succ, \Gamma)$ as a *limited consideration* model.

While various types of Γ can be considered depending on its interpretation, we mainly deal with consideration mappings obeying the following two properties. We say that Γ is an *attention filter (AF)*, if for every $A \in 2^X$ and $x \in A$,

$$x \notin \Gamma(A) \Longrightarrow \Gamma(A \backslash x) = \Gamma(A). \tag{1}$$

In words, Γ is an AF, if the consideration set is not affected when unrecognized alternatives are removed from a feasible set. On the other hand, we say that Γ is a *competition filter (CF)*,

[4]Classical textbook examples where backtracking is used are the eight queens puzzle, crossword puzzles and sudoku.

if for every $A' \subset A''$ and $x \in A'$,

$$x \notin \Gamma(A') \Longrightarrow x \notin \Gamma(A''). \tag{2}$$

That is, when Γ is a CF, if an alternative is not recognized in a smaller feasible set, then it cannot be recognized in a larger feasible set. When Γ satisfies both (1) and (2), then we say that Γ is a *competitive attention filter (CAF)*.[5] When Γ is an AF, a limited consideration model $(\succ, \Gamma)$ is referred to as an *AF-model*. Similarly, we say that $(\succ, \Gamma)$ is a *CF-model* (*CAF-model*), when Γ is a CF (CAF).

It is known that CF and CAF can be characterized as a result of filtering by coarse criteria $\{\mathcal{R}_i\}_{i\in I}$. Suppose that a DM has a family of binary relations $\{\mathcal{R}_i\}_{i\in I}$ and that as candidates for a choice, she picks up alternatives that are the best with respect to some $\mathcal{R}_i$.[6] Then, the consideration set $\Gamma(A)$ is represented as

$$\Gamma(A) = \bigcup_{i\in I}\{x \in A : x\mathcal{R}_i y \text{ for all } y \in A\backslash x\}. \tag{3}$$

The following proposition says that such a consideration mapping Γ is a CF, and conversely, any CF can be represented as (3) by using some family of binary relations $\{\mathcal{R}_i\}_{i\in I}$. In addition, if each $\mathcal{R}_i$ is assumed to be complete, asymmetric and transitive, then such filtering behavior characterizes a CAF. The former can be found, for example, in Lleras et al. (2017), while the latter is shown by Aizerman and Malishevski (1981).

Proposition A. *A consideration mapping $\Gamma : 2^X \to 2^X$ is a CF, if and only if there exists a set of binary relations $\{\mathcal{R}_i\}_{i\in I}$ with which Γ is represented as (3) for every $A \in 2^X$. In addition, Γ is a CAF, if and only if it is represented as (3) by using complete, asymmetric and transitive binary relations $\{\mathcal{R}_i\}_{i\in I}$.*

The preceding characterization motivates another important property of a CAF. Now, suppose that a DM has a set of criteria $\{\mathcal{R}_i\}_{i\in I}$ and some $A \in 2^X$ is feasible. It seems natural in

[5] We realize that these notions are considered in many papers, with being referred to as different names. We basically follow the terminologies in Masatlioglu et al. (2012) and Lleras et al. (2017). A survey by Moulin (1985) contains many theoretical results concerning these restrictions. For example, the statements in Propositions A – C below are essentially covered there.

[6] A decision theoretic analysis on such a two-step decision model, which is referred to as the *rationalization* model, is established by Cherepanov et al. (2013). Manzini and Mariotti (2012) proposed another type of two-step decision model called the *categorize-then-choose* model, which is known to be observationally equivalent to the rationalization model, though these models are conceptually quite distinct.

actual decision making that a DM makes a shortlist $\Gamma(A)$ sequentially, rather than calculating it at once. For instance, when A is divided into two parts, say, A' and A'', a DM may firstly derive $\Gamma(A')$ and then apply $\{\mathcal{R}_i\}_{i\in I}$ to $(\Gamma(A') \cup A'')$. Thus, this procedure will yield $\Gamma((\Gamma(A') \cup A'')$, while if a DM firstly looks at A'', then she will have $\Gamma(A' \cup \Gamma(A''))$. It is not difficult to check that if every $\mathcal{R}_i$ is complete, asymmetric and transitive, then $\Gamma(\Gamma(A') \cup A'') = \Gamma(A) = \Gamma(A' \cup \Gamma(A''))$. Thus, by Proposition A, if Γ is a CAF, then each $\Gamma(A)$ is robust to sequential derivation as above. Through an inductive argument, the robustness is valid even if a feasible set is divided into more than two parts. Moreover, by Aizerman and Malishevski (1981), the above property actually characterizes a CAF.

Proposition B. *A consideration mapping* Γ *is a CAF, if and only if for every* $A', A'' \in 2^X$, $\Gamma(A' \cup A'') = \Gamma(\Gamma(A') \cup A'')$, *which is referred to as path independence (PI).*

As an important special case of the above argued decision models, we refer to the *rational shortlist method* proposed by Manzini and Mariotti (2007). There, for each feasible set $A \in 2^X$, a consideration set is defined such that

$$\Gamma(A) = \{x \in A : \nexists x' \in A \text{ such that } x' \succ' x\}, \tag{4}$$

for some binary relation $\succ'$. That is, a DM only picks up undominated alternatives with respect to her first step preference. In Manzini and Mariotti (2007), a first step preference $\succ'$ is just assumed to be acyclic, while Au and Kawai (2011) deals with the case where $\succ'$ is asymmetric and transitive.[7] We say that Γ is a *(transitive) rational shortlist*, or in short, it is a *(T)RS*, if it can be described as (4) by using an acyclic (asymmetric and transitive) binary relation $\succ'$. By abuse of terminology, we refer to $(\succ, \Gamma)$ as a *(T)RS-model*, if Γ is a (T)RS. It is straightforward to check that RS is a special case of CF, and when $\succ'$ is asymmetric and transitive, which is a TRS, Γ defined in (4) is a CAF. Moreover, the following equivalence is known by Sen (1971) and Schwartz (1976).[8]

Proposition C. *A consideration mapping* $\Gamma : 2^X \to 2^X$ *is a (T)RS, if and only if it is a CF (CAF) and for every* $A', A'' \in 2^X$, $\Gamma(A') \cap \Gamma(A'') \subset \Gamma(A' \cup A'')$. *The latter property is referred*

[7]In Manzini and Mariotti (2007), they assumed that $\succ'$ is just asymmetric. However, since they also assume that the choice function is nonempty for all $A \in 2^X$, it is clear that $\succ'$ must be acyclic (otherwise $\Gamma(A)$ would be empty for some A). Note also that, in their original setting, a DM's preference $\succ$ is just asymmetric.

[8]By using it, one can confirm that amongst the examples (a) – (c) raised for CAF in Introduction, only (b) derives a TRS.

to the expansion axiom (EX).

3 Revealed preference tests

The purpose of this section is to provide a tool for testing each type of limited consideration models stated in the preceding section. Our analysis is based on a *data set* in the form of $\mathcal{O} = \{(a^t, A^t)\}_{t \in \mathcal{T}}$, where $\mathcal{T} = \{1, 2, ..., T\}$ is the set of indices of observations, $A^t \in 2^X$ is the feasible set at observation t, $a^t \in A^t$ is the chosen alternative at $t \in \mathcal{T}$, and $A^s \neq A^t$ is assumed for every $s \neq t$. Thus, for each observation point $t \in \mathcal{T}$, we observe a DM's choice and a feasible set, while a consideration set $\Gamma(A^t)$ is not observable. It should be stressed that following De Clippel and Rozen (2018a), we allow the case where an econometrician can observe a DM's choice behavior only on *some* feasible sets, rather than observing an entire choice function.

Given a data set as above, we would like to find a pair of strict preference and consideration mapping $(\succ, \Gamma)$ such that for each $t \in \mathcal{T}$, the observed choice a^t is the best alternative within $\Gamma(A^t)$ and that Γ obeys a specific restriction introduced in the preceding section (AF, CF, CAF and (T)RS).

Definition 1. A data set $\mathcal{O} = \{(a^t, A^t)\}_{t \in \mathcal{T}}$ is *rationalizable* by a limited consideration model $(\succ, \Gamma)$, if for every $t \in \mathcal{T}$, $a^t \in \Gamma(A^t)$ and $a^t \succ x$ for every $x \in \Gamma(A^t) \backslash a^t$. In particular, if $\mathcal{O}$ is rationalizable by a $\mathcal{P}$-model ($\mathcal{P}$ = AF, CF, CAF, (T)RS), then we say that $\mathcal{O}$ is $\mathcal{P}$-rationalizable.[9]

Note that when we say that $\mathcal{O}$ is $\mathcal{P}$-rationalizable, Γ is required to obey $\mathcal{P}$ on the entire domain 2^X, rather than on observed feasible sets. Obviously the relative strength of restrictions on Γ determines the relative strength of testing power of them: for example, if a data set is TRS-rationalizable, then it is also CAF-rationalizable, which in turn implies CF-rationalizability and AF-rationalizability.

Amongst five types of restrictions raised in the preceding section, tests for AF-model and CF-model are known in the literature: De Clippel and Rozen (2018a) established a test for $\mathcal{P}$ = AF, and a test for $\mathcal{P}$ = CF can be easily derived from Theorem 5 in Dean et al. (2017). The latter is proved in Appendix I.

[9] It is clear that any data set is rationalizable by some $(\succ, \Gamma)$ without any restriction on the shape of Γ. Indeed, we could just let $\Gamma(A^t) = \{a^t\}$ for every $t \in \mathcal{T}$, which does not work in general once some $\mathcal{P}$ (= AF, CF, CAF, (T)RS) is imposed.

Theorem A. A data set $\mathcal{O} = \{(a^t, A^t)\}_{t \in \mathcal{T}}$ is AF-rationalizable, if and only if there exists an acyclic binary relation $\succ^*$ such that for every $s, t \in \mathcal{T}$ with $a^s, a^t \in A^s \cap A^t$ and $a^s \neq a^t$,

$$\exists x \in A^s \backslash A^t \text{ such that } a^s \succ^* x \text{ or } \exists x \in A^t \backslash A^s \text{ such that } a^t \succ^* x. \tag{5}$$

Theorem B. A data set $\mathcal{O} = \{(a^t, A^t)\}_{t \in \mathcal{T}}$ is CF-rationalizable, if and only if the following binary relation $\succ^{\mathrm{CF}}$ is acyclic: for $x'' \neq x'$,

$$x'' \succ^{\mathrm{CF}} x', \text{ if for some } s, t \in \mathcal{T},\ x'' = a^s, x' = a^t \text{ and } \{x'', x'\} \subset A^s \subset A^t. \tag{6}$$

It may come as a surprise that tests for AF-model and CF-model have quite different structures. The test for CF-model stated in Theorem B has a "conventional" form of revealed preference tests in that it only requires testing acyclicity of a binary relation determined from a data set. On the other hand, in Theorem A, we have to directly check the existence of an acyclic binary relation with a specific property, to which De Clippel and Rozen (2018a) refers as an *acyclic satisfiability* problem. As shown in that paper, a certain class of acyclic satisfiability problems can be solved by using an efficient enumeration process, but Theorem A is not such a type of problem. Moreover, De Clippel and Rozen (2018a) proves that checking the condition in Theorem A is actually NP-hard.

In the rest of this section, we establish revealed preference tests for CAF-model and (T)RS-model, which are not provided in the literature. As we later show in an example, even if a data set simultaneously obeys the conditions in Theorems A and B, it may not be CAF-rationalizable, and hence, the test for CAF-model must be independently considered. It should be also noted that, as in Theorem A, revealed preference tests for these models are in the class of acyclic satisfiability problems. In addition, our tests also involve combinatorial calculations and applying them to actual data could be computationally challenging. Nevertheless, in our application, the help of a simple but powerful method called *backtracking* made the tests manageable including the test for AF-model based on Theorem A. See Appendix II for the basic idea of backtracking and how that method is adopted to our revealed preference tests.

3.1 A common starting point of tests

We start from a useful fact shared by all limited consideration models in this paper, and then proceed to a test for each specific model based on it.[10] Given a data set $\mathcal{O} = \{(a^t, A^t)\}_{t \in \mathcal{T}}$, define the *direct revealed preference* relation $\succ^R$ such that $x'' \succ^R x'$, if $x'' = a^t$ for some $t \in \mathcal{T}$, $x'' \neq x'$, and $x' \in A^t$. It is well known that a data set is consistent with the rational choice model, if and only if $\succ^R$ is acyclic, or the *strong axiom of revealed preference (SARP)* is satisfied. Put otherwise, if a data set $\mathcal{O}$ obeys SARP, then we can find a strict preference $\succ$ such that $(\succ, \Gamma)$ rationalizes $\mathcal{O}$ with Γ being the identity mapping. Since the identity mapping obeys all conditions concerning Γ referred to in this paper, our revealed preference tests become substantial when $\mathcal{O}$ contains revealed preference cycles.

A revealed preference cycle is formally defined as a set of pairs $\mathcal{C} = \{(x^k, x^{k+1})\}_{k=1}^K$ with $x^k \succ^R x^{k+1}$ for every $k = 1, 2, \ldots, K$, and $x^1 = x^{K+1}$. We refer to each (x^k, x^{k+1}) as an *arc* of a cycle. Now suppose that $\mathcal{O}$ is rationalizable by some limited consideration model, though it has revealed preference cycles. Then, since a DM has a strict preference $\succ$, for each cycle, there exists at least one arc (x^k, x^{k+1}) for which $x^{k+1} \succ x^k$. When there are Q revealed preference cycles in total, from each q-th cycle, pick up one of those arcs c_q to make a profile of arcs $\mathbf{D} = (c_1, c_2, ..., c_Q) \in \times_{q=1}^Q \mathcal{C}_q$. Note that, since each c_i is an ordered pair of components in X, $\mathbf{D}$ can be also regarded as a set of ordered pairs, or a binary relation on X. We interpret $\mathbf{D}$ as such whenever it is convenient. If a profile of arcs $\mathbf{D}$ is determined as above, it is effectively a part of a DM's strict preference $\succ$ in that $(x', x'') \in \mathbf{D}$ implies $x'' \succ x'$. Hence, $\mathbf{D}$ must be acyclic if it is regarded as a binary relation, and it also implies a connection between the above defined $\mathbf{D}$ and a DM's consideration mapping Γ as follows. For each $t \in \mathcal{T}$, define

$$B^t_{\mathbf{D}} = \{x \in A^t : (a^t, x) \in \mathbf{D}\}. \tag{7}$$

Then, every $x \in B^t_{\mathbf{D}}$ is available at A^t and preferred to a^t, which implies that $x \notin \Gamma(A^t)$. Put otherwise, $\Gamma(A^t) \subset A^t \backslash B^t_{\mathbf{D}}$ holds for every $t \in \mathcal{T}$. We summarize the above observation as a fact for future references.

Fact 1. *Suppose that a data set $\mathcal{O} = \{(a^t, A^t)\}_{t \in \mathcal{T}}$ is rationalizable by some limited consideration model and has Q revealed preference cycles. Then, there exists an acyclic selection of*

[10]In fact, tests for AF-model and CF-model can be also constructed through the idea provided below, which can be found in an earlier version of this paper, Inoue and Shirai (2018).

arcs from cycles $\mathbf{D} = (c_1, c_2, ..., c_Q) \in \times_{q=1}^{Q} \mathcal{C}_q$ *such that* $\Gamma(A^t) \subset A^t \backslash B^t_{\mathbf{D}}$ *for every* $t \in \mathcal{T}$*, where* $B^t_{\mathbf{D}} = \{x \in A^t : (a^t, x) \in \mathbf{D}\}$.

Note that the above fact is derived without using any specific property $\mathcal{P}$, and hence it is shared by any limited consideration model. On the other hand, once $\mathcal{P}$ is specified, we enhance the conclusion of Fact 1, which in turn derives a testable condition characterizing $\mathcal{P}$-rationalizability. This is exactly the strategy we take in the rest of this section.

3.2 Testing CAF-model

Here, we establish a revealed preference test for CAF-model. Suppose that a data set $\mathcal{O}$ with Q revealed preference cycles is CAF-rationalizable. To derive a characterization, by using the fact that Γ is a CAF, we strengthen Fact 1 step by step. In what follows, another expression of the definition of an AF in (1) is useful: for every $A, A' \in 2^X$, $\Gamma(A) \subset A' \subset A \Longrightarrow \Gamma(A') = \Gamma(A)$.

Fact 2. *Suppose that a data set* $\mathcal{O} = \{(a^t, A^t)\}_{t \in \mathcal{T}}$ *is CAF-rationalizable. Then, there exists an acyclic selection of arcs from cycles* $\mathbf{D} = (c_1, c_2, ..., c_Q) \in \times_{q=1}^{Q} \mathcal{C}_q$ *such that for every* $s, t \in \mathcal{T}$*,*

$$(A^s \backslash B^s_{\mathbf{D}}) \subset A^t \Longrightarrow \Gamma(A^t) \subset A^t \backslash B^s_{\mathbf{D}}. \tag{8}$$

Proof. Take any $\mathbf{D}$ for which $\Gamma(A^t) \subset A^t \backslash B^t_{\mathbf{D}}$ for every $t \in \mathcal{T}$, which exists by Fact 1. Note that when $(A^s \backslash B^s_{\mathbf{D}}) \subset A^t$ holds, we have $\Gamma(A^s) \subset A^s \backslash B^s_{\mathbf{D}} \subset (A^s \cap A^t) \subset A^s$. Then, since Γ is AF, $\Gamma(A^s) = \Gamma(A^s \cap A^t)$ must hold, which in turn implies that $x \in B^s_{\mathbf{D}} \Longrightarrow x \notin \Gamma(A^s \cap A^t)$. □

The above can be further extended as follows.

Fact 3. *Suppose that a data set* $\mathcal{O} = \{(a^t, A^t)\}_{t \in \mathcal{T}}$ *is CAF-rationalizable. Then, there exists an acyclic selection of arcs from cycles* $\mathbf{D} = (c_1, c_2, ..., c_Q) \in \times_{q=1}^{Q} \mathcal{C}_q$ *such that for every* $r, s, t \in \mathcal{T}$*,*

$$[(A^r \cup A^s) \backslash (B^r_{\mathbf{D}} \cup B^s_{\mathbf{D}})] \subset A^t \Longrightarrow \Gamma(A^t) \subset A^t \backslash (B^r_{\mathbf{D}} \cup B^s_{\mathbf{D}}). \tag{9}$$

Proof. Again, consider $\mathbf{D}$ obeying the property referred to in Fact 1. Then, both $\Gamma(A^r) \subset A^r \backslash B^r_{\mathbf{D}}$ and $\Gamma(A^s) \subset A^s \backslash B^s_{\mathbf{D}}$ hold. Since Γ is a CF, it holds that $x \in B^r_{\mathbf{D}} \Longrightarrow x \notin \Gamma(A^r \cup A^s)$ and $x \in B^s_{\mathbf{D}} \Longrightarrow x \notin \Gamma(A^r \cup A^s)$, which implies $\Gamma(A^r \cup A^s) \subset [(A^r \cup A^s) \backslash (B^r_{\mathbf{D}} \cup B^s_{\mathbf{D}})]$. Since $[(A^r \cup A^s) \backslash (B^r_{\mathbf{D}} \cup B^s_{\mathbf{D}})] \subset A^t$ is assumed, we have $[(A^r \cup A^s) \backslash (B^r_{\mathbf{D}} \cup B^s_{\mathbf{D}})] \subset [A^t \cap (A^r \cup A^s)] \subset (A^r \cup A^s)$. By the fact that Γ is an AF, it holds that $\Gamma(A^t \cap (A^r \cup A^s)) = \Gamma(A^r \cup A^s) \subset$

$[(A^r \cup A^s)\backslash(B^r_{\mathbf{D}} \cup B^s_{\mathbf{D}})]$. Finally, combining $[A^t \cap (A^r \cup A^s)] \subset A^t$ and Γ being a CF, we have $x \in (B^r_{\mathbf{D}} \cup B^s_{\mathbf{D}}) \Longrightarrow x \notin \Gamma(A^t)$ as desired. $\square$

By an inductive argument, ultimately we can extend (9) for any subset of indices $\tau \subset \mathcal{T}$ such that $\left(\bigcup_{r\in\tau} A^r \backslash \bigcup_{r\in\tau} B^r_{\mathbf{D}}\right) \subset A^t$. That is:

Fact 4. *Suppose that a data set $\mathcal{O} = \{(a^t, A^t)\}_{t\in\mathcal{T}}$ is CAF-rationalizable. Then, there exists an acyclic selection of arcs from cycles $\mathbf{D} = (c_1, c_2, ..., c_Q) \in \times_{q=1}^{Q}\mathcal{C}_q$ such that for every $\tau \subset \mathcal{T}$,*

$$\left(\bigcup_{r\in\tau} A^r \Big\backslash \bigcup_{r\in\tau} B^r_{\mathbf{D}}\right) \subset A^t \Longrightarrow \Gamma(A^t) \subset A^t \backslash \bigcup_{r\in\tau} B^r_{\mathbf{D}}. \tag{10}$$

The condition in Fact 4 depends on Γ, which is not observed by an econometrician, and hence we cannot directly check the existence of $\mathbf{D}$ obeying (10) from a data set. Nevertheless, we can convert it to a condition in terms of choices, which are observed in a data set. Indeed, Fact 4 implies that, when $\mathcal{O}$ is CAF-rationalizable, there must exist an acyclic section of arcs from cycles $\mathbf{D}$ such that $\left(\bigcup_{r\in\tau} A^r \Big\backslash \bigcup_{r\in\tau} B^r_{\mathbf{D}}\right) \subset A^t \Longrightarrow a^t \notin \bigcup_{r\in\tau} B^r_{\mathbf{D}}$. The right hand side follows, since $a^t \in \Gamma(A^t)$ must hold for every $t \in \mathcal{T}$.

CAF-condition: Suppose that a data set $\mathcal{O} = \{(a^t, A^t)\}_{t\in\mathcal{T}}$ contains Q revealed preference cycles. A selection of arcs from cycles $\mathbf{D} = (c_1, c_2, \ldots, c_Q) \in \times_{q=1}^{Q}\mathcal{C}_q$ obeys CAF-condition, if for every $t \in \mathcal{T}$ and any set of indices $\tau \subset \mathcal{T}$,

$$\left(\bigcup_{r\in\tau} A^r \Big\backslash \bigcup_{r\in\tau} B^r_{\mathbf{D}}\right) \subset A^t \Longrightarrow a^t \notin \bigcup_{r\in\tau} B^r_{\mathbf{D}}. \tag{11}$$

The existence of a selection of arcs obeying the above can be checked once a data set is given, and it is necessary for a data set to be CAF-rationalizable. More substantially, if we can find such an acyclic selection of arcs, then a data set is CAF-rationalizable. That is, CAF-rationalizability is tested by checking the existence of an acyclic selection of arcs from cycles obeying CAF-condition.

Theorem 1. *A data set $\mathcal{O} = \{(a^t, A^t)\}_{t\in\mathcal{T}}$ is CAF-rationalizable, if and only if there exists an acyclic selection of arcs from cycles that obeys CAF-condition.*

Remark. While the conditions look quite different, there is some similarity between the test for AF-model in Theorem A and the test for CAF-model in the preceding theorem. Namely, in both theorems, the conditions require the existence of acyclic binary relations with specific properties, that is, our test is also in the class of acyclic satisfiability. In addition, in both tests, acyclic binary relations in issue correspond to "partial guesses" of a DM's strict preference: in the proof of Theorem A given by De Clippel and Rozen (2018a), a binary relation $\succ^*$ works as a part of DM's preference relation, while, as argued in the preceding subsection, an acyclic selection of arcs $\mathbf{D}$ can be interpreted as a profile of "false" revealed preferences.

Gathered together with Propositions A and B in Section 2, testing CAF-model is equivalent to testing PI of DM's consideration as well as testing a two-step decision model with complete, asymmetric and transitive criteria.[11]

Corollary 1. *The following statements are equivalent:*

(i) *A data set $\mathcal{O}$ has an acyclic selection of arcs from cycles obeying CAF-condition.*

(ii) *A data set $\mathcal{O}$ is CAF-rationalizable.*

(iii) *A data set $\mathcal{O}$ is rationalizable by $(\succ, \Gamma)$ with Γ obeying PI.*

(iv) *A data set $\mathcal{O}$ is rationalizable by $(\succ, \Gamma)$ with Γ being expressed as (3) for a family of complete, asymmetric and transitive criteria $\{\mathcal{R}_i\}_{i\in I}$.*

Finally, we point out that the joint of the test for AF-model (Theorem A) and that for CF-model (Theorem B) does not work as a test for CAF-model. In the example below, a data set is AF-rationalizable and CF-rationalizable. However, it does *not* contain any acyclic selection of arcs from cycles obeying CAF-condition, or equivalently, it is not CAF-rationalizable.

Example 1. *Let $X = \{x_1, x_2, x_3\}$ and consider a data set of three observations as follows, where for each $t \in \mathcal{T}$, the chosen alternative is underlined:*

$$A^1 = \{\underline{x_1}, x_2\}, \quad A^2 = \{x_1, \underline{x_2}, x_3\}, \quad A^3 = \{x_2, \underline{x_3}\}.$$

This data set is AF-rationalizable and CF-rationalizable. To check the former, there are two pairs of observations, namely $(s,t) \in \{(1,2),(2,3)\}$ with $a^s, a^t \in A^s \cap A^t$ and $a^s \neq a^t$. Then, by letting $x_2 \succ^* x_1$ and $x_2 \succ^* x_3$, binary relation $\succ^*$ is acyclic and (5) is satisfied. For

[11]Recall that $\Gamma : 2^X \to 2^X$ obeys path independence (PI) if for every $A', A'' \in 2^X$, $\Gamma(A' \cup A'') = \Gamma((\Gamma(A') \cup A'')$.

example, it holds that for $(s,t) = (1,2)$, $x_3 \in A^2 \backslash A^1$ *and* $a^2 \succ^* x_3$. *Thus, this data set is AF-rationalizable. To see the latter, following (6) we have* $x_1 \succ^{CF} x_2$ *and* $x_3 \succ^{CF} x_2$. *Since* $\succ^{CF}$ *is acyclic, this data set is CF-rationalizable.*

However, this data set is not CAF-rationalizable. This data set contains two revealed preference cycles, namely $\mathcal{C}_1 : x_1 \succ^R x_2 \succ^R x_1$; $\mathcal{C}_2 : x_2 \succ^R x_3 \succ^R x_2$. *We firstly claim that any selection of arcs containing* (x_1, x_2) *cannot satisfy CAF-condition. Let* $\mathbf{D}$ *be a selection of arcs from cycles containing* (x_1, x_2). *Then,* $x_2 \in B^1_{\mathbf{D}}$, *and hence* $A^1 \backslash B^1_{\mathbf{D}} \subset A^2$ *and* $a^2 = x_2 \in B^1_{\mathbf{D}}$. *This is a violation of (11), and hence such a selection* $\mathbf{D}$ *cannot satisfy CAF-condition. Therefore, from* $\mathcal{C}_1$, *the arc* (x_2, x_1) *must be selected. Then, consider a selection of arcs* $\mathbf{D} = ((x_2, x_1), (x_2, x_3))$. *This derives* $B^2_{\mathbf{D}} = \{x_1, x_3\}$, *and we have* $A^2 \backslash B^2_{\mathbf{D}} = \{x_2\} \subset A^1$ *and* $a^1 = x_1 \in B^2_{\mathbf{D}}$, *which is a violation of (11). Thus, from* $\mathcal{C}_2$, *the arc* (x_3, x_2) *must be selected, and* $\mathbf{D} = ((x_2, x_1), (x_3, x_2))$ *is only one remaining possibility. This selection derives* $B^3_{\mathbf{D}} = \{x_2\}$ *and* $A^3 \backslash B^3_{\mathbf{D}} = \{x_3\} \subset A^2$. *However, since* $a^2 = x_2 \in B^3_{\mathbf{D}}$, *this* $\mathbf{D}$ *also violates CAF-condition.*

3.3 Testing (T)RS-model

Then, we turn to the case of (T)RS-model. Based on the nature of the model, we can extend Fact 1 as follows. Suppose that $\mathcal{O} = \{(a^t, A^t)\}_{t \in \mathcal{T}}$ contains Q revealed preference cycles and is (T)RS-rationalizable. Then, a DM has two preferences $\succ'$ and $\succ$, where the former is acyclic (asymmetric and transitive) while the latter is a strict preference, and the consideration mapping Γ is defined as (4). Then, Fact 1 implies that there exists an acyclic selection of arcs from cycles $\mathbf{D} = (c_1, c_2, \dots, c_Q)$ such that the corresponding $\{B^t_{\mathbf{D}}\}_{t \in \mathcal{T}}$ satisfies $\Gamma(A^t) \subset A^t \backslash B^t_{\mathbf{D}}$ for every $t \in \mathcal{T}$. Since a DM obeys a (T)RS-model, for every $x' \in B^t_{\mathbf{D}}$, there exists some $x'' \in A^t \backslash x'$ such that $x'' \succ' x'$. This in turn implies that x' is not considered as long as x'' is feasible, and hence, $x' \succ^R x''$ is impossible.

Given the discussion above, we can define a binary relation $\rhd$ on X such that $x'' \rhd x'$ if $x' \in B^t_{\mathbf{D}}$ for some $t \in \mathcal{T}$, $x'' \in A^t \backslash x'$, and $x' \not\succ^R x''$. Since we start from a data set consistent with a (T)RS-model, for every $x' \in B^t_{\mathbf{D}}$, there exists at least one $x'' \in A^t \backslash x'$ with $x'' \rhd x'$ for which $x'' \succ' x'$ actually holds. Loosely speaking, $\rhd$ can be seen as a broad guess of the first step preference $\succ'$. In addition, the acyclicity of $\succ'$ requires that we can always find a selection $\rhd' \subset \rhd$ that is acyclic, and for every $t \in \mathcal{T}$ and $x' \in B^t_{\mathbf{D}}$, there exists some $x'' \in A^t \backslash x'$ with $x'' \rhd' x'$. Furthermore, if the first step preference $\succ'$ is assumed to be transitive, a selection $\rhd'$

has to be chosen so that

$$\text{for every } x' \in B^t_{\mathbf{D}} \text{ and } z^1, ..., z^k,\ x'' \rhd' z^1 \rhd' \cdots \rhd' z^k \rhd' x' \Longrightarrow x' \not\succ^R x''. \tag{12}$$

Now, $\rhd'$ is a "correct" guess of the first step preference, and if transitivity is imposed, the above implies that $x'' \succ' x'$. Hence, if $x' \succ^R x''$ were to hold, then it leads to a contradiction that x' is deleted from a consideration set from which it is actually chosen. In fact, this observation is summarized in the conditions below, and plays a key role to characterize a data set that is rationalizable by a (T)RS-model.

(T)RS-condition: Suppose that a data set $\mathcal{O} = \{(a^t, A^t)\}_{t\in\mathcal{T}}$ contains Q revealed preference cycles. An acyclic selection of arcs from cycles $\mathbf{D} = (c_1, c_2, \ldots, c_Q) \in \times_{q=1}^{Q} \mathcal{C}_q$ obeys RS-condition, if for the corresponding $\{B^t_{\mathbf{D}}\}_{t\in\mathcal{T}}$, there exists an acyclic selection $\rhd'$ of $\rhd$, where for every $t \in \mathcal{T}$,

$$\text{for every } x' \in B^t_{\mathbf{D}}, \text{ there exists } x'' \in A^t \text{ with } x'' \rhd' x'. \tag{13}$$

When $\rhd'$ can be chosen so that (12) is also satisfied, we say that $\mathbf{D}$ obeys TRS-condition.

Theorem 2. *A data set $\mathcal{O} = \{(a^t, A^t)\}_{t\in\mathcal{T}}$ is (T)RS-rationalizable, if and only if there exists an acyclic selection of arcs from cycles obeying (T)RS-condition.*

Remark. As seen from the statement, testing (T)RS-model is also in the class of acyclic satisfiability. In De Clippel and Rozen (2018a), they showed that testing RS-model is NP-hard, while they did not show a revealed preference test for that model. Note that, as argued there, it is possible to show the computational complexity even if the condition is not exactly specified.

Given Proposition C in Section 2, testing (T)RS-model is equivalent to testing the joint of CF (CAF) and EX.[12]

Corollary 2. *The following statements are equivalent:*

(i) A data set $\mathcal{O}$ has an acyclic selection of arcs from cycles obeying (T)RS-condition.

[12] Recall that a consideration mapping Γ obeys the expansion axiom (EX), if for every $A', A'' \in 2^X$, $\Gamma(A') \cap \Gamma(A'') \subset \Gamma(A' \cup A'')$.

(ii) A data set $\mathcal{O}$ is (T)RS-rationalizable.

(iii) A data set $\mathcal{O}$ is rationalizable by $(\succ, \Gamma)$ with Γ being a CF (CAF) that obeys EX.

Finally, the following example shows that RS-model has a strictly stronger observable restriction than CF-model. Similarly, one can construct an example that is CAF-rationalizable, but not TRS-rationalizable.

Example 1. (continued) *Reconsider the data set in the preceding subsection:*

$$A^1 = \{\underline{x_1}, x_2\}, \quad A^2 = \{x_1, \underline{x_2}, x_3\}, \quad A^3 = \{x_2, \underline{x_3}\},$$

where chosen alternatives are underlined. While this data set is CF-rationalizable, it is not RS-rationalizable. We firstly claim that any selection of arcs containing (x_1, x_2) or (x_3, x_2) cannot satisfy RS-condition. Whenever (x_1, x_2) is contained in $\mathbf{D}$, we have $x_2 \in B^1_{\mathbf{D}}$. Meanwhile, since there is no $x \in A^1$ such that $x_2 \not\succ^R x$, it is impossible to define $\rhd$ so that x_2 is dominated by an alternative in A^1, which violates (13). A parallel logic shows that (x_3, x_2) cannot be an arc selected from $\mathcal{C}_2$. Hence $\mathbf{D} = ((x_2, x_1), (x_2, x_3))$ is the only remaining possibility. However, in this case, $B^2_{\mathbf{D}} = \{x_1, x_3\}$, and thus $x_1 \rhd x_3$ and $x_3 \rhd x_1$ hold, and there does not exist an acyclic selection of $\rhd$ that obeys (13).

4 Simulation and Experiment

Given the theorems in the preceding section, now we can test AF, CF, CAF and (T)RS-models from a data set $\mathcal{O} = \{(a^t, A^t)\}_{t \in \mathcal{T}}$. In this section, we apply the tests for these models both to randomly generated data sets and experimental data sets in order to deal with the following two issues: one is to compare relative strength of observable restrictions across models based on randomly generated data sets, and the other is to compare the measure of "plausibility" across models based on experimental data sets.

The former can be regarded as a version of Bronars' test in the context of limited consideration models, and one can measure the strength of observable restriction of each model by using its pass rate.[13] If we collect a sufficiently large number of random choices according to a uniform distribution, then the pass rate approximates the proportion of choices that are

[13]Bronars (1987) deals with the revealed preference test of the classical consumer theory. There, the fail rate of GARP on randomly generated data sets on randomly generated budgets is calculated.

model-consistent to all logically possible choices. If this value is very close to 1, then the model in question is very hard to refute, or its observable restriction is weak.

As shown by Selten (1991) and Beatty and Crawford (2011), this measure of observable restriction plays a key role in considering the measure of plausibility of a model based on empirical or experimental data sets, which is nothing but our second issue in this section. Given empirical or experimental data sets, *Selten's index* evaluates a model by the difference of the pass rate calculated from actual data sets and the proportion of model-consistent choices to all logically possible choices. Practically, as in Beatty and Crawford (2011), Selten's index is calculated as the difference between pass rate based on actual data and that of randomly generated data sets. We could say that a model with a higher Selten's index is "better" than that with a lower Selten's index, or intuitively, a "nice" model in terms of Selten's index is a model with higher pass rate and stronger observable restrictions.[14]

4.1 Simulation

We generated 10,000 random data sets with $|X| = 10, |\mathcal{T}| = 20, \min |A^t| = 2$, and $\max |A^t| = 8$. Firstly, we randomly generated 100 profiles of feasible sets $\mathbb{A}_n := \{A_n^t\}_{t\in\mathcal{T}}$ for $n = 1, \ldots, 100$: fixing n, in generating each A_n^t, we set $|A_n^t| \in \{2, \ldots, 8\}$ following a uniform distribution over the set of natural numbers $\{2, \ldots, 8\}$, and then choose $|A_n^t|$ elements from X following a uniform distribution over X. We also require that $A_n^s \neq A_n^t$ for $s \neq t$. For each profile of feasible sets $\mathbb{A}_n = \{A_n^t\}_{t\in\mathcal{T}}$, a profile of choices $\{a_{i,n}^t\}_{t\in\mathcal{T}}$ is randomly generated for $i = 1, \ldots, 100$: fixing n and i, $a_{i,n}^t$ is chosen following a uniform distribution over A_n^t for every $t \in \mathcal{T}$. Consequently we have a random choice data set $\mathcal{O}_{i,n} = \{(a_{i,n}^t, A_n^t)\}_{t\in\mathcal{T}}$ for $i = 1, \ldots, 100$, to which we apply our revealed preference tests. Note that we randomize feasible sets, as well as choices over them, since in general, observable restriction of a specific model depends on the structure of the feasible sets $\mathbb{A}_n = \{A_n^t\}_{t\in\mathcal{T}}$. For example, if $A^s \cap A^t = \varnothing$ for every $s, t \in \mathcal{T}$, then SARP is trivially satisfied, which implies that all five limited consideration models are non-refutable. For each $\mathcal{O}_{i,n} = \{(a_{i,n}^t, A_n^t)\}_{t\in\mathcal{T}}$, we tested AF, CF, CAF and (T)RS-models, as well as SARP.

[14] Selten's index has an axiomatization as follows. Let $m(\alpha, \beta) \in [-1, 1]$ be a measure of plausibility of a model that depends on the empirical pass rate, say, $\alpha \in (0, 1)$, and the proportion of model-consistent choices to all logically possible choices, say, $\beta \in (0, 1)$. Then, any $m(\cdot, \cdot)$ obeying the following axioms is an affine transformation of Selten's index $\alpha - \beta$: [MONOTONICITY] $m(1, 0) > m(0, 1)$, [EQUIVALENCE] $m(1, 1) = m(0, 0)$, and [AGGREGABILITY] $m(\lambda\alpha_1 + (1-\lambda)\alpha_2, \lambda\beta_1 + (1-\lambda)\beta_2) = \lambda m(\alpha_1, \beta_1) + (1-\lambda)m(\alpha_2, \beta_2)$. The first and second axioms determine how a measure should deal with extreme realizations of α and β, and the third axiom essentially implies that $m(\cdot, \cdot)$ is a cardinal measure.

test	pass rates
SARP	0
AF	0.9927
CF	0.6298
CAF	0.0396
RS	0.0259
TRS	0.0006

Table 1: Average pass rates.

In Table 1, the average pass rates of 100 different profiles of feasible sets (10,000 data sets) are summarized.[15] It shows that AF-model is extremely permissive, letting more than 99% of the random data sets pass the test, and CF-model is also quite permissive. On the other hand, we can say that observable restrictions of CAF and (T)RS-models are reasonably strong. What is striking is that, while more than 60% of all data sets pass both tests for AF-model and CF-model, the pass rate of CAF-model is significantly lower (lower than 4%).[16] Similarly, the difference between RS and TRS is also huge, where TRS is the conjunction of RS and AF by Proposition C. Thus, although the hypothesis of Γ being an AF is very hard to reject by itself, combining it with some other restrictions could strengthen observable restrictions drastically. As another viewpoint, if we interpret Γ as a result of filtering by some criteria $\{\mathcal{R}_i\}_{i\in I}$, by Proposition A, the difference between CF and CAF-models comes from the additional assumption that every $\mathcal{R}_i$ is complete, asymmetric and transitive in the latter. Similarly, by Proposition C, the difference between CF (CAF) and RS (TRS) is explained by whether Γ obeys EX or not.

Finally, in Figure 1, we visually summarize the distributions of pass rates for each test, where the horizontal axis is the pass rate given the profile of feasible sets, running from 0 to 1 with bin width 0.05. The vertical axis is the frequency of profiles of feasible sets of which the pass rates drop in each bin. It shows that the pass rate for the test for CF-model has a large variance depending on the structure of feasible sets, while pass rates for other models

[15] In Table 1, the pass rate of SARP is zero. While in theory there exist choice patterns consistent with it, there were none within our samples.

[16] Randomly generated data sets can be partitioned into eight types: (i) TRS-rationalizable; (ii) RS-rationalizable and CAF-rationalizable but not TRS-rationalizable; (iii) RS-rationalizable but not CAF-rationalizable; (iv) CAF-rationalizable but not RS-rationalizable; (v) both CF-rationalizable and AF-rationalizable but neither CAF nor RS-rationalizable; (vi) only CF-rationalizable; (vii) only AF-rationalizable; (viii) none of these types. Out of 10,000 data sets, the distribution of types is as follows: (i) 24 data sets, (ii) 4 data sets, (iii) 186 data sets, (iv) 369 data sets, (v) 5575 data sets, (vi) 140 data sets, (vii) 3576 data sets and (viii) 126 data sets.

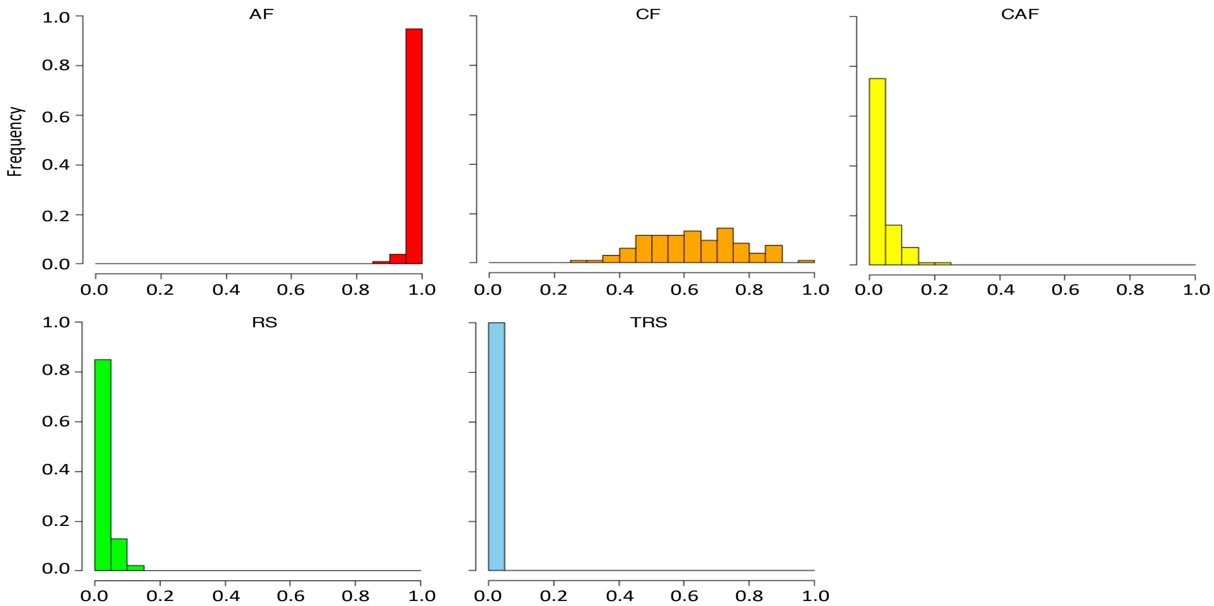


Figure 1: Histograms.

are more accumulated around either 0 or 1.

4.2 Experiment

We now proceed to the second issue of this section, or the comparison of Selten's indices based on experimental data sets. Amongst the profiles of feasible sets $\{\mathbb{A}_n\}_{n=1}^{100}$ generated for simulation, we chose one of them: one where the pass rates of the five limited consideration models are fairly "balanced." In choosing one profile of feasible sets, we first listed several of them where (i) pass rate of the test for AF-model is not 1 and (ii) pass rates of the tests for most models were distinct. Then for each of these profiles, we generated 1000 random choices, in order to assess the pass rates of each model in further detail. Finally, we picked one profile of feasible sets where pass rates of all five models were distinct, pass rate of the test for AF-model is not 1, and that of TRS-model is not 0.

Following the experimental design of Manzini and Mariotti (2009), we consider the situation where each subject chooses remuneration plans of installments.[17] Each remuneration plan consists of 2400 Japanese yen overall, and this amount is split and installed in one month,

[17]Note that Manzini and Mariotti (2009) is the working paper version of Manzini and Mariotti (2012).

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}
in 1 month	450	800	1150	450	450	800	850	1200	1550	500
in 3 months	800	800	800	450	1500	1150	0	0	0	0
in 5 months	1150	800	450	1500	450	450	1550	1200	850	1900

Table 2: The remuneration plans (in Japanese yen).

three months, and five months after the experiment is conducted. Since the profile of feasible sets is one of those used in simulation, there are 10 alternatives and 20 feasible sets in total, and each feasible set consists of 2 – 8 alternatives (see Appendix III for the contents of each feasible set). These numbers are not known to subjects. Most of the ten alternatives are in line with the eight alternatives used in Manzini and Mariotti (2009): there are "increasing," "constant," "decreasing" and "jump" series of payments and we added two "hump" payments in order to make the total number of alternatives ten. The alternatives are listed in Table 2: alternatives x_1 and x_7 are "increasing," x_2 and x_8 are "constant," x_3 and x_9 are "decreasing," x_4 and x_{10} are "jump," and x_5 and x_6 are "hump" series.[18]

The pass rate for each model is in the left column of Table 3. Statistical differences between the pass rates are significant at SARP & TRS (at 1% significance level); AF & CF (5%); AF & CAF (1%); and CAF & RS (1%) in terms of a two-sample t-test assuming equal variance. The center column indicates random pass rates of models; to derive them, we generated 500,000 random choices over the feasible sets. Then, Selten's index of each model is derived as the difference between pass rates of experimental data and randomly generated data: for example Selten's index of CAF-model is $0.8832 = 0.9115 - 0.0283$. Looking at Table 3, for this experimental setting, CAF-model distinctively well-performs in terms of Selten's index, while its pass rate is not significantly different from CF-model.[19]

[18]The experiment was carried out at an experimental economics laboratory at the Faculty of Political Science and Economics, Waseda University, Japan. We ran 4 sessions and there were a total of 113 subjects. Subjects were recruited through an on-line bulletin that is accessible by all students. The proportion of male and female subjects were roughly the same. The experiment was computerized using the experimental software z-Tree by Fischbacher (2007), and each participant was seated individually with a separator so that they cannot look at other participants' choices. Experimental sessions lasted an average of 42 minutes, of which the average duration of effective play was 11 minutes. The shortest session lasted 36 minutes and the longest 51 minutes. At the beginning of the experiment, subjects read instructions on paper, while the experimenter read the instructions aloud (see Appendix III for an English translated version of the instructions). Preceding the remuneration-relevant stages, subjects were asked to take part in practice stages in order to be familiar with the usage of the computer in the experiment, and all subjects had to correctly answer questions that were asked to check whether the subjects understood the experimental design. It was explained that at the end of the experiment, one screen would be selected at random, and the chosen remuneration plan at that screen would be actually installed.

[19]Concerning the random pass rate of SARP, similar to the case of Table 1, in theory there exist choices that are

	experiment pass rates	random pass rates	Selten's index
SARP	0.3363	0.0000	0.3363
AF	1.0000	0.9639	0.0361
CF	0.9558	0.4714	0.4844
CAF	0.9115	0.0283	0.8832
RS	0.5929	0.0157	0.5772
TRS	0.5841	0.0012	0.5829

Table 3: Experimental pass rates, random pass rates, and Selten's indices.

Remark. Concerning the experimental pass rates, we also test whether there are statistical differences between pass rates of subjects when we partition them with respect to decision time, sex and reason of decision, using a two-sample t-test allowing different variances. There is no statistical difference between long-decision-time subjects and short-decision-time subjects; no statistical difference between male and female.[20] In a questionnaire following the experiment, we showed the subjects three experiment screens with their actual chosen alternatives indicated, and asked reasons of their choices. There are two clusters of subjects whose answers were consistent across these three decisions: one is a cluster of subjects who wanted to receive money "as soon as possible (a.s.a.p)" (29 subjects), and the other is a cluster of subjects who would like to receive money "as equally as possible through the three installments (smoothing)" (15 subjects). One may suspect these subjects tend to be more rational, but there are no statistical differences between these 44 subjects and the others; the "a.s.a.p." subjects and the others; the "smoothing" subjects and the others; or the "a.s.a.p." subjects and the "smoothing" subjects.

We finally refer to the result of a comparative experiment, in which the cardinality of feasible sets varies from 2 to 5. The set of alternatives is the same as the baseline setting, and the number of feasible sets is also the same as before. The purpose of this experiment is to see how the size of feasible sets affects the comparison in terms of Selten's index. Similar to the case of the baseline setting, we fix 20 feasible sets so that pass rates of five models are distinct, and pass rate of the test for AF-model is not 1 and that of TRS is not 0. This experiment was carried out at the same facility with the baseline experiment, and there were a total of 80 subjects in 3 sessions.[21]

The experimental pass rates, random pass rates, and Selten's indices are summarized in

consistent with SARP, but there were none within the 500,000 randomly generated choices.

[20] A long-decision-time (short-decision-time) subject is a subject whose average decision time across 20 decisions

	experiment pass rates	random pass rates	Selten's index
SARP	0.3875	0.0000	0.3875
AF	1.0000	0.9718	0.0282
CF	0.9375	0.7987	0.1388
CAF	0.9125	0.3674	0.5451
RS	0.6250	0.2038	0.4212
TRS	0.6125	0.0623	0.5502

Table 4: Result of the comparative experiment.

Table 4. Comparing the experimental pass rates of Tables 3 and 4, it seems that the pass rates of relatively rational models, namely RS, TRS and SARP, are slightly higher in the comparative setting, where subjects choose from smaller feasible sets. However, there was no statistical significance in the difference of pass rates of each model across the two experiments. On the other hand, random pass rates are quite different across the two experiments, and hence Selten's indices differ as well. In this comparative experiment, we see that TRS-model explains subjects' behavior the best, and the explanatory power of CAF is not as high as in the baseline experiment, due to the fact that the random pass rate is higher in this comparative experiment. This shows that the explanatory power of models may vary drastically in different environments, even when the observed pass rates are similar.

Appendix I: Proofs

Proof of Theorem B

Let $\succ^*$ be a linear extension of $\succ^{\text{CF}}$, and define Γ such that

$$\Gamma(A) = \left[\left(\bigcup_{t:A^t \supset A} \{a^t\}\right) \cap A\right] \cup \{x \in A : y \succ^* x \text{ for all } y \in A \backslash x\}. \tag{14}$$

Then $\Gamma(A) \neq \varnothing$ for every nonempty $A \in 2^X$, and $a^t \in \Gamma(A^t)$ for all $t \in \mathcal{T}$. Moreover, by definition of $\succ^{\text{CF}}$, a^t is the best alternative in $\Gamma(A^t)$ in terms of $\succ^*$. The remaining issue is whether the above defined Γ is a CF, which can be confirmed as follows. Let $\bar{x} \in A' \subset A''$,

is longer (shorter) than the median of all subjects.

[21]The recruitment procedure of subjects is also same with the baseline experiment, and no subject in this comparative experiment participated in the baseline experiment, and vice versa.

and let $\bar{x} \in \Gamma(A'')$. If $\bar{x} \in \{x \in A'' : y >^* x \text{ for all } y \in A''\backslash x\}$, then $\bar{x}$ is the worst alternative (w.r.t. $>^*$) in a larger set A'', and hence it must be also the worst alternative in A'. If $\bar{x} \in \left[\left(\bigcup_{t:A^t \supset A''}\{a^t\}\right) \cap A''\right]$, then it is obvious that $\bar{x} \in \left[\left(\bigcup_{t:A^t \supset A'}\{a^t\}\right) \cap A'\right]$ also holds. In both cases, it holds that $\bar{x} \in \Gamma(A')$, which implies that Γ is a CF.

Proof of Theorem 1

We construct a pair of consideration mapping and strict preference that rationalizes $\mathcal{O}$ based on an acyclic selection of arcs from cycles $\mathbf{D}$ (and the corresponding $\{B^t_{\mathbf{D}}\}_{t \in \mathcal{T}}$) obeying CAF-condition. To define Γ, we need the following set of indices for every $A \in 2^X$:

$$\tau(A) = \max\left\{\tau \subset \mathcal{T} : \bigcup_{r \in \tau} A^r \Big\backslash \bigcup_{r \in \tau} B^r_{\mathbf{D}} \subset A\right\}. \tag{15}$$

Then, by using $\tau(A)$, define Γ such that

$$\Gamma(A) = A \Big\backslash \bigcup_{r \in \tau(A)} B^r_{\mathbf{D}}. \tag{16}$$

Obviously, in order for the above definition to be well-defined, $\tau(A)$ must be uniquely determined for every $A \in 2^X$, which is actually the case. To see this, suppose to the contrary: there exist $\tau_1(A) \neq \tau_2(A)$ that obey (15). Then, we have $\left(\bigcup_{r \in \tau_1(A)} A^r \backslash \bigcup_{r \in \tau_1(A)} B^r_{\mathbf{D}}\right) \subset A$ and $\left(\bigcup_{r \in \tau_2(A)} A^r \backslash \bigcup_{r \in \tau_2(A)} B^r_{\mathbf{D}}\right) \subset A$, which implies that

$$\left[\bigcup_{r \in \tau_1(A) \cup \tau_2(A)} A^r \Big\backslash \left(\bigcup_{r \in \tau_1(A)} B^r_{\mathbf{D}} \cup \bigcup_{r \in \tau_2(A)} B^r_{\mathbf{D}}\right)\right] \subset A.$$

Obviously, this can be rewritten as

$$\left(\bigcup_{r \in \tau_1(A) \cup \tau_2(A)} A^r \Big\backslash \bigcup_{r \in \tau_1(A) \cup \tau_2(A)} B^r_{\mathbf{D}}\right) \subset A.$$

By defining $\tau(A) = \tau_1(A) \cup \tau_2(A)$, we have $\tau(A) \supsetneq \tau_i(A)$ for $i = 1, 2$, which contradicts the maximality of $\tau_1(A)$ and $\tau_2(A)$.

Given that Γ defined as (16) is well-defined, we move on to show that Γ is both AF and CF. Consider any $A', A'' \in 2^X$ with $A' \subset A''$, and $x \in A'$ such that $x \notin \Gamma(A')$. This means that

$x \in \bigcup_{r\in\tau(A')} B^r_{\mathbf{D}}$. Since $\tau(\cdot)$ is clearly monotonic, it follows that $\tau(A') \subset \tau(A'')$, and hence, $x \in \bigcup_{r\in\tau(A'')} B^r_{\mathbf{D}}$. This assures that $x \notin \Gamma(A'')$, which shows that Γ is CF. To see AF, take any $A \in 2^X$ and any $x \in A$ with $x \notin \Gamma(A)$. This means that $x \in \bigcup_{r\in\tau(A)} B^r_{\mathbf{D}}$, which in turn implies

$$\left(\bigcup_{r\in\tau(A)} A^r \Big\backslash \bigcup_{r\in\tau(A)} B^r_{\mathbf{D}}\right) \subset A\backslash x. \tag{17}$$

The maximality and uniqueness of $\tau(\cdot)$, combined with (17), imply $\tau(A) \subset \tau(A\backslash x)$. On the other hand, the monotonicity of $\tau(\cdot)$ implies $\tau(A\backslash x) \subset \tau(A)$. Hence we have $\tau(A) = \tau(A\backslash x)$. Then, $\Gamma(A\backslash x) = (A\backslash x)\backslash \bigcup_{r\in\tau(A\backslash x)} B^r_{\mathbf{D}} = A\backslash \bigcup_{r\in\tau(A)} B^r_{\mathbf{D}} = \Gamma(A)$, which shows that Γ is AF.

Let $\succ^*$ be a binary relation such that $x'' \succ^* x'$, if $x'' = a^t$, $x' \in \Gamma(A^t)$, and $x'' \neq x'$. We show that $\succ^*$ is acyclic, and thus extendable to a strict preference. By way of contradiction, suppose that there exists a cycle $x^1 \succ^* x^2 \succ^* \cdots \succ^* x^L \succ^* x^1$, which clearly implies $x^1 \succ^R x^2 \succ^R \cdots \succ^R x^L \succ^R x^1$. Then, there exists an arc $(x^\ell, x^{\ell+1})$ contained in $\mathbf{D}$. Since $x^\ell = a^t$ and $x^{\ell+1} \in A^t$ hold for some $t \in \mathcal{T}$, this means that $x^{\ell+1} \in B^t_{\mathbf{D}}$ for such an observation t. It is easy to check from the definition of Γ that $t \in \tau(A^t)$, and hence, $x^{\ell+1} \notin \Gamma(A^t) \subset A^t\backslash B^t_{\mathbf{D}}$. However, then, it holds that $x^\ell \not\succ^* x^{\ell+1}$, which is a contradiction.

Finally, let us show that $a^t \in \Gamma(A^t)$ for every $t \in \mathcal{T}$, which follows immediately from CAF-condition. Indeed, for every $t \in \mathcal{T}$, we have $\left(\bigcup_{r\in\tau(A^t)} A^r \backslash \bigcup_{r\in\tau(A^t)} B^r_{\mathbf{D}}\right) \subset A^t$, and then, CAF-condition requires $a^t \notin \bigcup_{r\in\tau(A^t)} B^r_{\mathbf{D}}$, which in turn ensures $a^t \in \Gamma(A)$ for every $t \in \mathcal{T}$. Since $\succ^*$ is acyclic, it is extendable to a strict preference $\succ$ on X using Szpilrajn's theorem. Then this $\succ$ and Γ defined as (16) combined together is a CAF-model that rationalizes $\mathcal{O}$.

Proof of Theorem 2

The proofs for RS-model and TRS-model are almost identical, so we provide the proofs of them jointly. Since the necessity parts of them have been already discussed, we prove the sufficiency parts of them based on an acyclic selection of arcs from cycles obeying (T)RS-condition. Using an acyclic selection $\rhd'$ of $\rhd$, define Γ as

$$\Gamma(A) = \{x \in A : \nexists x' \in A \text{ such that } x' \rhd' x\}. \tag{18}$$

Note that the selection $\rhd'$ is acyclic, so we use it as a first step preference for RS-model. If we can find $\rhd'$ so that it obeys (12) in addition to (13), then we use the transitive closure of it, say, $\rhd''$ as a first step preference and define Γ by using it instead of $\rhd'$. This $\rhd''$ works as a first step preference for TRS-model. Note further that $\Gamma(A^t) \subset A^t \backslash B^t_{\mathbf{D}}$ holds, by the definition of $\rhd'$ (or $\rhd''$) and the construction of Γ. The remaining substantial parts of the proof are to show that $a^t \in \Gamma(A^t)$ for every $t \in \mathcal{T}$, and the binary relation $\succ^*$ defined as $x'' \succ^* x'$ if $x'' = a^t, x' \in \Gamma(A^t)$, and $x'' \neq x'$ is acyclic.

To prove that $\succ^*$ is acyclic, suppose to the contrary, i.e., there is a cycle: $x^1 \succ^* x^2 \succ^* \cdots \succ^* x^L \succ^* x^1$. Since we have $\succ^* \subset \succ^R$, this cycle implies $x^1 \succ^R x^2 \succ^R \cdots \succ^R x^L \succ^R x^1$. Then, it must be the case that there exists an arc $(x^\ell, x^{\ell+1})$ contained in $\mathbf{D}$, and we have $x^{\ell+1} \in B^t_{\mathbf{D}}$ for every $t \in \mathcal{T}$ with $x^\ell = a^t$ and $x^{\ell+1} \in A^t$. By (T)RS-condition, there exists some $x \in A^t$ such that $x \rhd' (\rhd'') x^{\ell+1}$, which in turn implies $x^{\ell+1} \notin \Gamma(A^t)$. Then it is impossible to have $x^\ell = a^t \succ^* x^{\ell+1}$, and we conclude that $\succ^*$ is acyclic.

To see that $a^t \in \Gamma(A^t)$ for every $t \in \mathcal{T}$, by way of contradiction, suppose that for some $t \in \mathcal{T}$, $a^t \notin \Gamma(A^t)$. This means that there exists $x \in A^t \backslash a^t$ such that $x \rhd' a^t$, which in turn implies $x \rhd a^t$. However, this is not possible, since $x \rhd a^t$ requires $a^t \not\succ^R x$, while we have $a^t \succ^R x$. When $\mathbf{D}$ obeys TRS-condition and Γ is defined as the set of maximal elements with respect to $\rhd''$, $a^t \notin \Gamma(A^t)$ implies the existence of some $x \in A^t \backslash a^t$ such that $x \rhd'' a^t$. However, this is also impossible, since $x \rhd'' a^t$ implies the existence of a sequence $z^1, z^2, ..., z^k$ such that $x \rhd' z^1 \rhd' \cdots \rhd' z^k \rhd' a^t$, and by TRS-condition, $a^t \not\succ^R x$, which contradicts the assumption that $x \in A^t$. The rest of the proof is to extend the transitive closure of $\succ^*$ to a strict preference by using Szpilrajn's theorem. Then it can easily be seen that the data set is rationalized by a (T)RS-model $(\succ, \Gamma)$.

Appendix II: Backtracking

The revealed preference tests for CAF-model and (T)RS-model involve combinatorial calculations, and applying them to actual data may be computationally challenging. However, the tests become manageable with the help of a simple but powerful method called *backtracking*.[22] Here we illustrate how this method is adopted to our revealed preference tests, after a brief introduction of this method. Note that the method here is also applicable to Theorem A by De Clippel and Rozen (2018a), and we actually employ the algorithm here in our data analysis.

[22] Some foundational references of the backtracking method are Walker (1960), Davis, Logemann and Loveland (1962), and Golomb and Baumert (1965).

To get the basic idea of backtracking, consider a problem where we have to select c_q from some set $\mathcal{C}_q$ for every $q = 1, 2, ..., Q$, so that the resulting selection $(c_1, c_2, \ldots, c_Q)$ obeys some constraint $\mathbf{P}_Q$. While there are $\prod_{q=1}^{Q} |\mathcal{C}_q|$ logically possible trials that we must check, the backtracking procedure may lead us to a solution with much fewer trials, especially when $\mathbf{P}_Q$ has the *cut-off* property defined below. For every $\bar{Q} < Q$, let us refer to $(c_1, c_2, \ldots, c_{\bar{Q}})$ as a *partial* selection in the sense that c_q is not yet determined for $q \in \{\bar{Q} + 1, \ldots, Q\}$. Then, we say that $\mathbf{P}_Q$ has the cut-off property if: (I) for every $\bar{Q} < Q$, there exists a constraint $\mathbf{P}_{\bar{Q}}$, which is a length-$\bar{Q}$-modified version of $\mathbf{P}_Q$; and (II) partial selection $(c'_1, c'_2, \ldots, c'_{\bar{Q}})$ violating $\mathbf{P}_{\bar{Q}}$ implies violation of $\mathbf{P}_{\bar{Q}+1}$ for any partial selection $(c'_1, c'_2, \ldots, c'_{\bar{Q}}, c_{\bar{Q}+1})$. Given the cut-off property, if some partial selection $(c'_1, c'_2, \ldots, c'_{\bar{Q}})$ violates $\mathbf{P}_{\bar{Q}}$, then there is no need to waste time on searching for subsequent components $c_{\bar{Q}+1}, \ldots, c_Q$, since there is no chance of any selection $(c'_1, c'_2, \ldots, c'_{\bar{Q}}, c_{\bar{Q}+1}, \ldots, c_Q)$ satisfying $\mathbf{P}_Q$. In fact, this feature is at the heart of backtracking, and allows us to adopt a component-by-component search for a desired selection.

Given below is a basic algorithm of the backtracking method. We consider a case where $\mathcal{C}_q$ is finite for every q, so with no loss of generality, we assume that sets $\mathcal{C}_q$ are a sets of integers.

Basic backtracking algorithm. Given sets $(\mathcal{C}_q)_{q=1}^{Q}$ and constraints $(\mathbf{P}_q)_{q=1}^{Q}$, this algorithm yields a selection $(c_1, c_2, \ldots, c_Q)$ that satisfies $\mathbf{P}_Q$, or $\varnothing$ (meaning that $\mathbf{P}_Q$ cannot be satisfied).

1. [Initialize.] Set $\bar{Q} \leftarrow 0$.
2. [Enter level $\bar{Q}+1$.] (Now $(c_1, \ldots, c_{\bar{Q}})$ obeys $\mathbf{P}_{\bar{Q}}$.) Set $\bar{Q} \leftarrow \bar{Q}+1$. Then set $c_{\bar{Q}} \leftarrow \min \mathcal{C}_{\bar{Q}}$.
3. [Test $(c_1, \ldots, c_{\bar{Q}})$.] If $(c_1, \ldots, c_{\bar{Q}})$ obeys $\mathbf{P}_{\bar{Q}}$, go to 6.
4. [Try again.] If $c_{\bar{Q}} \neq \max \mathcal{C}_{\bar{Q}}$, set $c_{\bar{Q}}$ to the next larger element of $\mathcal{C}_{\bar{Q}}$, and go to 3.
5. [Backtrack.] Set $c_{\bar{Q}} \leftarrow \min \mathcal{C}_{\bar{Q}}$ and $\bar{Q} \leftarrow \bar{Q} - 1$. If $\bar{Q} = 0$, return $\varnothing$ and stop. Otherwise, go to 4.
6. [Terminate.] If $\bar{Q} = Q$, return $(c_1, \ldots, c_{\bar{Q}})$ and stop. Otherwise, go to 2.

The big picture of this algorithm is as follows. The process initially starts from considering a singleton selection (c_1) and sees whether $\mathbf{P}_1$ is satisfied. If there is no such element in $\mathcal{C}_1$, then we can immediately conclude that there is no chance of finding a selection $(c_1, c_2, \ldots, c_Q)$ obeying $\mathbf{P}_Q$. If we find a successful partial selection $(c_1, c_2, \ldots, c_{\bar{Q}-1})$ and reach the $\bar{Q}$-th level, we set $c_{\bar{Q}}$ to be the minimum element in $\mathcal{C}_{\bar{Q}}$, and test whether $(c_1, c_2, \ldots, c_{\bar{Q}})$ obeys $\mathbf{P}_{\bar{Q}}$. If $\mathbf{P}_{\bar{Q}}$ is satisfied, then we proceed to the $(\bar{Q} + 1)$-th level. If not, we redefine $c_{\bar{Q}}$ to be the next larger element of $\mathcal{C}_{\bar{Q}}$ and check $\mathbf{P}_{\bar{Q}}$. If we cannot find any $c_{\bar{Q}} \in \mathcal{C}_{\bar{Q}}$ such that $(c_1, c_2, \ldots, c_{\bar{Q}})$ obeys $\mathbf{P}_{\bar{Q}}$, then we go back to the $(\bar{Q}-1)$-th level and update $c_{\bar{Q}-1}$. This search algorithm terminates when we succeed in finding some $(c_1, c_2, \ldots, c_Q)$ obeying $\mathbf{P}_Q$, or it is determined that any (partial) selection with $c_1 = \max C_1$ cannot be successful.

We now show that the backtracking method is applicable to our revealed preference tests as follows. Suppose that a data set $\mathcal{O} = \{(a^t, A^t)\}_{t\in\mathcal{T}}$ has Q revealed preference cycles. For each $q = 1, 2, ..., Q$, let $\mathcal{C}_q$ be the q-th revealed preference cycle. Then, for every $\mathcal{P} \in$ {CAF, RS, TRS}, if we set $\mathbf{P}_Q$ as the joint of acyclicity and $\mathcal{P}$-condition, the revealed preference test for $\mathcal{P}$-model is equivalent to the existence problem of a selection of arcs from cycles $\mathbf{D} = (c_1, c_2, ..., c_Q)$ obeying constraint $\mathbf{P}_Q$. We claim that the above defined $\mathbf{P}_Q$ obeys the cut-off property for every $\mathcal{P} \in$ {CAF, RS, TRS}.

Condition (I): We define $\mathbf{P}_{\bar{Q}}$ for every $\bar{Q} \leqslant Q$ as follows. Given a partial selection of arcs $\mathbf{D}_{\bar{Q}} = (c_1, c_2, \ldots, c_{\bar{Q}})$, note that $\mathbf{D}_{\bar{Q}}$ can be regarded as a binary relation. Therefore, acyclicity is a well-defined constraint. Now we define a partial sequence version of $\mathcal{P}$-condition, to which we refer as $\mathcal{P}_{\bar{Q}}$-condition as follows. Similar to (7), we can define for every $t \in \mathcal{T}$,

$$B^t_{\bar{Q}} = \{x \in A^t : (a^t, x) \in \mathbf{D}_{\bar{Q}}\}. \tag{19}$$

We say that a partial selection of arcs from cycles $\mathbf{D}_{\bar{Q}} = (c_1, c_2, ..., c_{\bar{Q}})$ obeys $\mathcal{P}_{\bar{Q}}$-condition, if the corresponding $\{B^t_{\bar{Q}}\}_{t\in\mathcal{T}}$ satisfies the restriction in $\mathcal{P}$-condition; specifically, $\mathbf{D}_{\bar{Q}}$ obeys CAF$_{\bar{Q}}$-condition, if it holds that for every $t \in \mathcal{T}$ and any set of indices $\tau \subset \mathcal{T}$,

$$\left(\bigcup_{r\in\tau} A^r \backslash \bigcup_{r\in\tau} B^r_{\bar{Q}}\right) \subset A^t \Longrightarrow a^t \notin \bigcup_{r\in\tau} B^r_{\bar{Q}}. \tag{20}$$

Similar terminology is used for other models as well. With this $\mathcal{P}_{\bar{Q}}$-condition, we let $\mathbf{P}_{\bar{Q}}$ be the joint of acyclicity and $\mathcal{P}_{\bar{Q}}$-condition, which is clearly a well-defined constraint.

Condition (II): We show that if a partial selection of arcs from cycles $\mathbf{D}_{\bar{Q}} = (c_1, c_2, ..., c_{\bar{Q}})$ does not satisfy $\mathbf{P}_{\bar{Q}}$ for some $\bar{Q} < Q$, then $\mathbf{D}_{\bar{Q}+1} = (c_1, c_2, ..., c_{\bar{Q}}, c_{\bar{Q}+1})$ cannot satisfy $\mathbf{P}_{\bar{Q}+1}$ for any $c_{\bar{Q}+1} \in \mathcal{C}_{\bar{Q}+1}$. It is obvious, if $\mathbf{D}_{\bar{Q}}$ is cyclic, then $\mathbf{D}_{\bar{Q}+1}$ cannot be acyclic. Therefore, the substantial part is $\mathcal{P}_{\bar{Q}}$-condition. However, this follows straightforwardly by taking a look at our revealed preference conditions and the construction of B^t-sets, which is shown below.

Fact 5. *If a partial selection of arcs from cycles* $\mathbf{D}_{\bar{Q}} = (c_1, c_2, ..., c_{\bar{Q}})$ *fails* $\mathcal{P}_{\bar{Q}}$*-condition, then partial selection of arcs from cycles* $\mathbf{D}_{\bar{Q}+1} = (c_1, c_2, ..., c_{\bar{Q}}, c_{\bar{Q}+1})$ *fails* $\mathcal{P}_{\bar{Q}+1}$*-condition.*

Proof. Note that the selection of arcs from cycles $1, 2, \dots, \bar{Q}$ are the same in $\mathbf{D}_{\bar{Q}}$ and $\mathbf{D}_{\bar{Q}+1}$. Hence, it follows from (19) that $B^t_{\bar{Q}} \subset B^t_{\bar{Q}+1}$ for every $t \in \mathcal{T}$. By the structure of $\mathcal{P}_{\bar{Q}}$-condition and $\mathcal{P}_{\bar{Q}+1}$-condition, we can see the following: whenever we have "larger" B^t-sets, (i) the LHS of CAF-condition is more permissive and (ii) $\rhd$ is stronger and thus more difficult to find an acyclic (asymmetric and transitive) selection of it in (T)RS-condition. Both (i) and (ii) imply that $\mathbf{D}_{\bar{Q}+1}$ fails $\mathcal{P}_{\bar{Q}+1}$-condition whenever $\mathbf{D}_{\bar{Q}}$ fails $\mathcal{P}_{\bar{Q}}$-condition. □

Example 2. *Let* $X = \{x_1, x_2, x_3, x_4, x_5, x_6\}$ *and consider a data set of six observations as follows, where for each* $t \in \mathcal{T}$*, the chosen alternative is underlined:*

$$A^1 = \{\underline{x_1}, x_2, x_4\}, \quad A^2 = \{x_1, \underline{x_2}\}, \quad A^3 = \{\underline{x_3}, x_4, x_6\},$$
$$A^4 = \{x_3, \underline{x_4}\}, \quad A^5 = \{x_2, \underline{x_5}, x_6\}, \quad A^6 = \{x_5, \underline{x_6}\}.$$

Let us walk through the backtracking algorithm, and see how we determine that the data set is not rationalizable by an RS-model. Note that the data set has four cycles (we order the cycles and the arcs in them as below):

1. $\mathcal{C}_1 = \{(x_1, x_2), (x_2, x_1)\}$,
2. $\mathcal{C}_2 = \{(x_3, x_4), (x_4, x_3)\}$,
3. $\mathcal{C}_3 = \{(x_5, x_6), (x_6, x_5)\}$,
4. $\mathcal{C}_4 = \{(x_1, x_4), (x_4, x_3), (x_3, x_6), (x_6, x_5), (x_5, x_2), (x_2, x_1)\}$.

For every $\bar{Q} \in \{1, 2, 3, 4\}$*, let us denote by* $\mathbf{P}_{\bar{Q}}$ *the joint of acyclicity and RS*$_{\bar{Q}}$*-condition. Following our backtracking procedure, we first set* $\bar{Q} = 1$ *and set* $c_1 = (x_1, x_2)$*, which is the first arc of the first cycle. Since single element selection* $((x_1, x_2))$ *obeys* $\mathbf{P}_1$*, we proceed to the second cycle by setting* $\bar{Q} = 2$*. Here we set* $c_2 = (x_3, x_4)$ *and check whether* $((x_1, x_2), (x_3, x_4))$ *obeys* $\mathbf{P}_2$*, which is affirmative. Then we go to the third cycle by setting* $\bar{Q} = 3$ *and set* $c_3 = (x_5, x_6)$*. In fact, this partial selection of arcs from cycles* $((x_1, x_2), (x_3, x_4), (x_5, x_6))$ *fails to satisfy* $\mathbf{P}_3$*, specifically RS*$_3$*-condition. In this case, we keep* $\bar{Q} = 3$*, and update* c_3 *to the next arc in* $\mathcal{C}_3$*, and set* $c_3 = (x_6, x_5)$*. Then, we test whether this updated selection of arcs* $((x_1, x_2), (x_3, x_4), (x_6, x_5))$ *obeys* $\mathbf{P}_3$*, which is negative. At this point, we can determine that it is impossible to find a selection* (c_1, c_2, c_3, c_4) *obeying RS-condition and acyclicity as long as*

$\bar{Q}$	(partial) selection	$\mathbf{P}_{\bar{Q}}$
1	$((x_1, x_2))$	PASS
2	$((x_1, x_2), (x_3, x_4))$	PASS
3	$((x_1, x_2), (x_3, x_4), (x_5, x_6))$	FAIL
3	$((x_1, x_2), (x_3, x_4), (x_6, x_5))$	FAIL
2	$((x_1, x_2), (x_4, x_3))$	FAIL
1	$((x_2, x_1))$	FAIL
0	$\varnothing$	**STOP**

Table 5: Backtracking procedure applied to Example 2 for testing RS-model.

$(x_1, x_2), (x_3, x_4)$ *are selected from* $\mathcal{C}_1, \mathcal{C}_2$ *respectively. Thus we backtrack* $\bar{Q}$ *to 2, and update* c_2 *to* (x_4, x_3)*. Looking at* $((x_1, x_2), (x_4, x_3))$*, it fails* $\mathbf{P}_2$*. Since there is no chance of success unless* (x_1, x_2) *is discarded from the selection, we rewind* $\bar{Q}$ *to 1, and update* c_1 *to* (x_2, x_1)*. Then we check whether* $((x_2, x_1))$ *obeys* $\mathbf{P}_1$*, which is negative. Then* $\bar{Q}$ *is set to 0 and the algorithm terminates, which means that the data set is not rationalizable by RS-model.*

Remark 1: One advantage of the backtracking approach is that we may be able to determine, at an early stage of the process of search, that a data set fails the test. Due to this feature, calculation time does depend on how we order the cycles. We suggest that the cycles are sorted so that shorter cycles come first: whenever $q' < q''$, q'-th cycle is weakly shorter than q''-th cycle. The cycles in Example 2 are sorted in this way. Whenever this takes too much calculation time, it seems natural to list "problematic" cycles first. Problematic cycles are those such that a (partial) selection of arcs from cycles fails when adding an arc at that cycle. This may allow us to determine that a data set fails the test at an early stage of the backtracking process (and we actually adopt this type of strategy).

Remark 2: Backtracking can be applied to De Clippel and Rozen (2018a)'s AF-test as well. Recall that their test requires the existence of an acyclic binary relation $\succ^*$ such that, for every $s, t \in \mathcal{T}$, with $a^s, a^t \in A^s \cap A^t$ and $a^s \neq a^t$,

$$\exists x \in A^s \backslash A^t : a^s \succ^* x \text{ or } \exists x \in A^t \backslash A^s : a^t \succ^* x.$$

Suppose there are $Q > 0$ pairs of observations (s, t) such that $a^s, a^t \in A^s \cap A^t$ and $a^s \neq a^t$. It can be seen that backtracking is applicable to De Clippel and Rozen's test, by letting $\mathbf{P}_Q$ be acyclicity, and for q-th pair (s,t), defining

$$\mathcal{C}_q = \left\{(x'', x') : [x'' = a^s \text{ and } x' \in A^s \backslash A^t] \text{ or } [x'' = a^t \text{ and } x' \in A^t \backslash A^s]\right\}.$$

Appendix III: Experimental setting

Here we provide the following materials regarding the experiment: (i) the structure of feasible sets used in our baseline and comparative experiment, (ii) an example of a computer screen used in the experiment, and (iii) an English translated version of the experimental instructions. In the tables of feasible sets below, alternative x_i corresponds to the i-th remuneration plan in Table 4. Note that in the actual experiment, the order of feasible sets was randomized as well as the order of alternatives on the screen, so as to avoid framing effects. Tables 6 and 7 are lists of the twenty feasible sets that we used in the baseline and comparative experiments respectively. The rows correspond to indices of observation $t \in \{1, 2, \ldots, 20\}$ and columns correspond to alternatives: the (i, j)-th entry is 1 if and only if $x_j \in A^i$.

Figure 2 is an example of a computer screen used in the experiment. This feasible set consists of four alternatives, and the subject chooses an alternative by clicking the "Choose this" button corresponding to it. Once this button is clicked, the subject has to confirm her choice, in order to minimize the possibility of errors. The last three pages of this appendix are allocated to an English translated version of the instructions used in our experiment.

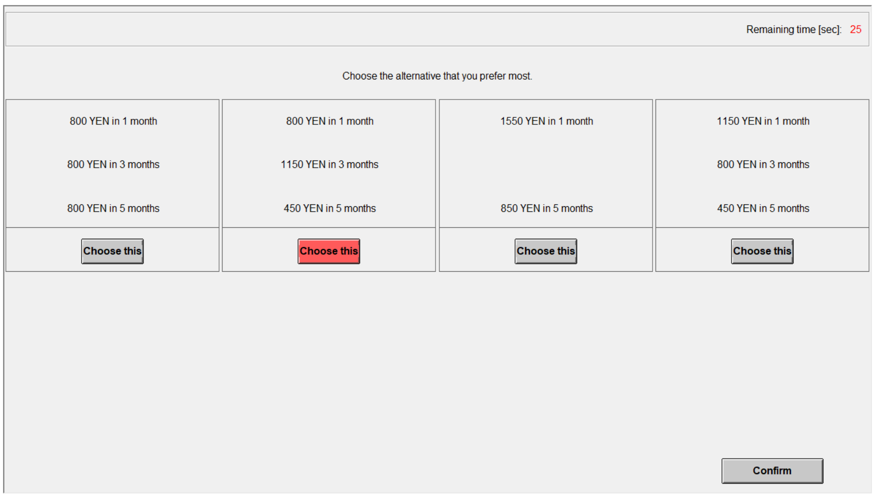


Figure 2: An example of PC screen.

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}
1	1	1	0	0	0	0	0	0	0	0
2	0	1	0	1	0	0	0	0	0	0
3	0	1	0	0	0	0	0	1	0	0
4	0	0	1	0	0	1	0	0	0	0
5	0	0	0	0	1	0	0	1	0	0
6	0	0	0	0	1	0	0	0	1	0
7	1	0	0	0	1	1	0	0	0	0
8	0	0	0	0	1	0	1	1	0	0
9	1	1	1	0	0	0	0	0	1	0
10	1	0	1	0	1	1	0	0	0	0
11	0	1	1	0	0	1	0	0	1	0
12	1	0	1	0	0	1	1	1	0	0
13	1	0	0	0	0	1	1	1	0	1
14	1	1	1	0	0	1	1	1	0	0
15	1	1	0	0	1	1	1	0	0	1
16	1	0	1	0	1	1	0	0	1	1
17	1	1	1	0	1	1	1	0	0	1
18	1	1	1	0	1	0	1	1	0	1
19	1	0	1	0	1	1	0	1	1	1
20	0	1	0	1	1	1	1	1	1	1

Table 6: Feasible sets of baseline experiment.

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}
1	0	1	0	0	0	1	0	0	0	0
2	0	0	1	0	0	0	0	0	1	0
3	0	0	0	1	0	1	0	0	0	0
4	0	0	0	0	1	1	0	0	0	0
5	0	0	0	0	0	1	1	0	0	0
6	0	0	0	0	0	0	1	0	0	1
7	1	1	0	0	0	0	0	0	1	0
8	0	0	1	0	1	1	0	0	0	0
9	0	0	1	0	0	1	0	0	1	0
10	0	0	0	1	1	0	0	0	0	1
11	0	0	0	0	1	1	0	1	0	0
12	0	0	0	0	0	0	1	1	1	0
13	1	0	0	1	1	0	0	0	1	0
14	0	0	0	1	1	0	1	1	0	0
15	1	0	0	1	1	1	0	0	0	1
16	1	0	0	1	0	1	1	0	1	0
17	0	1	1	1	1	0	0	0	1	0
18	0	1	1	0	1	1	0	0	1	0
19	0	1	1	0	0	1	0	0	1	1
20	0	0	0	1	0	1	1	0	1	1

Table 7: Feasible sets of comparative experiment.

Instructions

<u>NOTE</u>: **You are not allowed to communicate with others during the experiment.**

You are not allowed to use your mobile phone or any other personal electronic device.

Everyone is given the same instruction. This experiment is conducted to study individual decision-making behavior. This project is funded by KAKENHI (Japan Society for the Promotion of Science).

１．Outline of the experiment

In the experiment, you will repeatedly face some question screens. Each question screen will display several alternatives regarding "how to split and receive 2400 JPY in 1 month, 3 months, and 5 months after the experiment." While the amount of money you receive in 1 month, 3 months, and 5 months respectively differs depending on which alternative you choose, the total amount of payment is 2400 JPY for every alternative. At each question screen, you are asked to choose the alternative that you prefer most.

２．Example of experiment question screen

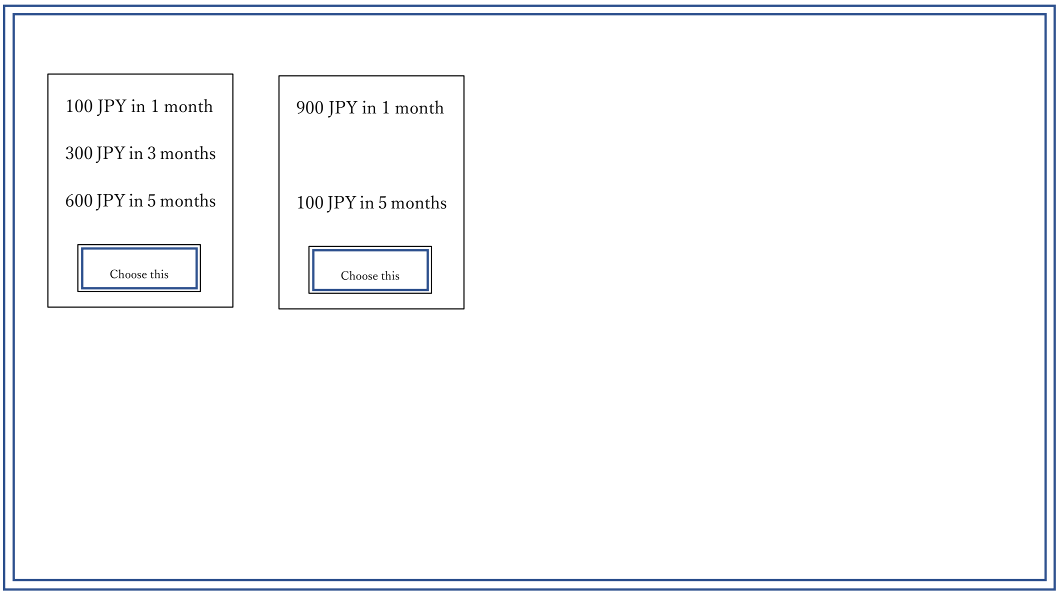
In order to get an idea of how alternatives will appear in the experiment, we show below some examples. In the actual experiment, the overall amount of money you receive is 2400 JPY, but in this hypothetical example the total amount is fixed to 1000 JPY. The example screens are shown in page 3.

Figure PC screen (a) displays two alternatives: the left and right alternative. In the left alternative, 100 JPY will be installed in 1 month, 300 JPY in 3 months, and 600 JPY in 5 months, and in the right alternative, 900 JPY will be installed in 1 month and 100 JPY will be installed in 5 months. If you prefer the left alternative, click the "Choose this" button on the left. If you prefer the right alternative, click the "Choose this" button on the right.

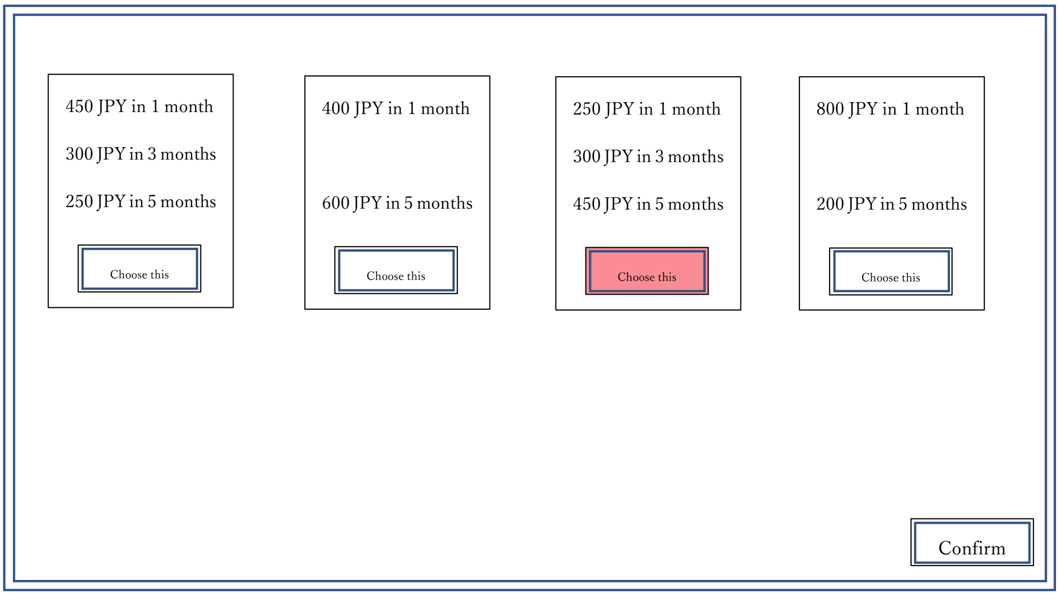
Figure PC screen (b) displays four alternatives. In this screen, the “Choose this” button of the second from the right alternative (250 JPY will be installed in 1 month, 300 JPY in 3 months, and 450 JPY in 5 months) is clicked. If you want to confirm this choice, then click the “Confirm” button that appears on the bottom left of the screen. You can re-select other alternatives until this “Confirm” button is clicked.

3. About payment method

In the experiment, you will be repeatedly asked to choose an alternative from question screens as in page 3. At the end of the experiment, one of the question screens will be chosen at random. Then according to the alternative that you chose at that question screen, 2400 JPY will be split and installed in 1 month, 3 months, and 5 months after the experiment. The payments will be installed by Waseda University, to the bank account that you registered to the university.



PC screen (a)



PC screen (b)

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