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Limited consideration and limited data: revealed preference tests and observable restrictions

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Abstract

This paper develops revealed preference tests for choice models under limited consideration, allowing a partially observed data set. Leading theories in the literature such as the limited attention model, the rationalization model, the categorize-then-choose model, and the rational shortlisting models are covered. Given a tool for testing limited consideration models, we analyze the empirical aspects of them. Our revealed preference tests are applied to randomly generated data sets to compare the strength of observable restrictions across various models. In addition, we carried out an experiment to compare models in terms of Selten's index, which is a measure for plausibility of a model in explaining a given data set. As a result, remarkable differences are seen both in observable restrictions and Selten's indices across models.

KEYWORDS: Revealed preference; Limited consideration; Limited attention; Rational shortlisting; Bronars' test; Selten's index

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1 Introduction

Let X be a finite set of alternatives, and $A \subset X$ be a set of feasible alternatives for an agent. Following the classical choice theory, an agent will choose the most preferred alternative according to her preference which is often assumed to be a strict preference. In testing if an agent's behavior can be accounted for by this standard framework, the theory of revealed preference is one of the most prevailing methods for economists. Typically, we collect finitely many observations of an agent's behavior $\mathcal{O} = \{(a^t, A^t)\}_{t \in \mathcal{T}}$, where $\mathcal{T}$ is the set of indices of observations, $A^t \subset X$ is the set of feasible alternatives at observation t, and a^t is the chosen alternative from A^t. It is well known that a data set is consistent with the standard choice framework, if and only if it obeys the *strong axiom of revealed preference (SARP)*, which requires acyclicity of the direct revealed preference $\succ^R$ defined as $x'' \succ^R x'$, if $x'' = a^t$ for some $t \in \mathcal{T}$, $x'' \neq x'$, and $x' \in A^t$.

However, as pointed out in a number of experimental studies, violation of SARP is not rare at all, and various theories of bounded rationality have been proposed for systematic analyses of cyclical choices. Amongst others, in the recent decision theory literature, a class of decision procedures so called *limited consideration models* has been widely studied. There, some feasible alternatives are a priori excluded from an agent's consideration due to the limitation of recognition capacity and/or due to the shortlisting according to some criteria different from her preference (e.g., psychological restrictions, a preference on categories rather than alternatives, and others). As a result, for each feasible set A, an agent maximizes her preference relation not necessarily on A itself, but on some subset $\Gamma(A) \subset A$, which we call a *consideration set.* The primal objective of this paper is to develop the counterparts of SARP for this type of decision models. That is, given a data set $\mathcal{O} = \{(a^t, A^t)\}_{t \in \mathcal{T}}$, we provide a necessary and sufficient condition under which $\mathcal{O}$ is consistent with some specific type of limited consideration model in the following sense: we can find a strict preference $\succ$ and $\Gamma(\cdot)$ such that for every $t \in \mathcal{T}$, $a^t \succ x$ whenever $x \in \Gamma(A^t) \backslash a^t$.

It is clear that, without any restriction on a set mapping Γ, testing a limited consideration model is vacuous in that any choice behavior is accounted for by letting $\{a^t\} = \Gamma(A^t)$ for every $t \in \mathcal{T}$. Thus, we deal with models where some restrictions *are* imposed on an agent's *consideration mapping* $\Gamma : 2^X \to 2^X$, which specifies her consideration set for every $A \subset X$. Possible restrictions include the following two types which are well-established in the literature: (i) the *attention filter property (AFP)*, which requires that for every $A \subset X$ and $x \in A$,

$x \notin \Gamma(A) \implies \Gamma(A \backslash x) = \Gamma(A)$ and (ii) the *competition filter property (CFP)*, which requires that for every $A' \subset A''$, $x \notin \Gamma(A') \implies x \notin \Gamma(A'')$. In words, AFP requires that the removal of unrecognized alternatives does not change the set of recognized alternatives, while CFP requires that an alternative ignored at a smaller feasible set cannot be recognized at a larger feasible set. If we adopt both AFP and CFP as plausible restrictions, then it seems natural to consider the joint of them, or (iii) Γ obeying both AFP and CFP, to which we refer as AFP+CFP. In addition, as an important special case of Γ obeying CFP, we take into account (iv) the *rational shortlist method (RSM)* in Manzini and Mariotti (2007). There, an agent makes a shortlist consisting of maximal elements with respect to an asymmetric first step preference, and then she makes a choice to maximize her preference relation. For every $A \subset X$, a shortlist as above can be captured as the consideration set $\Gamma(A)$. A stronger version of RSM is also considered, namely (v) the *transitive rational shortlist method (TRSM)*, where a first step preference is asymmetric and transitive. In this model, Γ obeys AFP+CFP and even more. We show a necessary and sufficient condition under which a data set $\mathcal{O} = \{(a^t, A^t)\}_{t \in \mathcal{T}}$ is consistent with each of the above five models.

In the literature of decision theory, revealed preference characterizations for these models have been provided in terms of a *choice function*, which is equivalent to a data set with $\{A^t\}_{t \in \mathcal{T}} = 2^X$. For example, Masatlioglu, Nakajima, and Ozbay (2012) and Lleras, Masatlioglu, Nakajima, and Ozbay (2015, 2017) characterize AFP, CFP, and AFP+CFP models in terms of restrictions on a choice function.[1] Choice functions derived from RSM and TRSM models are respectively characterized by Manzini and Mariotti (2007) and Au and Kawai (2011). On the other hand, these characterizations are mainly for clarifying the normative aspects of models rather than testing them from actual data sets. Indeed, they are *not* straightforwardly extendable to the case of a general data set $\mathcal{O} = \{(a^t, A^t)\}_{t \in \mathcal{T}}$. De Clippel and Rozen (2014) is the first paper to shed light on revealed preference *tests* for limited consideration models, where a test for AFP models based on a general data set is provided. This paper can be regarded as a follow-up of their paper in that we add revealed preference tests for other important models, as well as simulation and experimental data analyses. Meanwhile, it should be noted that, as explained in Sections 2 and 3, our approach for testing models is distinct from that of De Clippel and Rozen, and even for AFP models, our necessary and sufficient

[1] Lleras, Masatlioglu, Nakajima, and Ozbay (2015) is a working paper version of Lleras, Masatlioglu, Nakajima, and Ozbay (2017).

condition takes a form quite different from their condition.

Given tools for testing limited consideration models, we apply them to look at the empirical aspects of the models. One possible direction is comparison of the relative restrictiveness between models. It is obvious that limited consideration models are relatively permissive compared to the rational choice model, and it is known that there are several subclass/superclass relations within limited consideration models. Meanwhile, it is not at all clear *how* relatively restrictive/permissive the models are. This can be assessed through a simulation. Following Bronars (1987), we generate random choices and apply our tests to see the fraction of data that are consistent with each model. Provided that observable restriction of each model depends on the structure of feasible sets, we generate random profiles of feasible sets as well as choices upon them. In our simulation, we stick to the environment with 20 feasible sets each of which contains 2 – 8 alternatives out of 10 alternatives. The result is rather striking: the strength of observable restriction is quite different across models. AFP model is very hard to reject with average pass rate of random data exceeding 99%, and CFP model is also permissive with average pass rate exceeding 60%. However, the joint of them, or AFP+CFP model, is far more restrictive with average pass rate being less than 4%. Thus, the joint of rather weak behavioral restrictions could result in strong observable restrictions. The rational shortlisting type models both have strong testing power: the average pass rate of RSM is less than 3% and that of TRSM is less than 0.1%.

Furthermore, we carried out an experiment to compare models in terms of *Selten's index*, which is a measure for evaluating a model as an explanation of data. Given choice data of subjects, Selten's index of a model is practically calculated as the difference between pass rate of the revealed preference test of actual data and that of randomly generated choices. Loosely speaking, a model is highly evaluated if (i) it can well explain observed choices, while (ii) its observable restriction is strong (see Selten (1991) and Beatty and Crawford (2011)). In our baseline experiment, we adopted one profile of feasible sets generated in the simulation part, i.e., each subject was asked to make choices on 20 feasible sets containing 2 – 8 alternatives out of 10 alternatives. Amongst 113 subjects, 33% of them passed SARP, about 60% were consistent with RSM/TRSM, and the pass rates for AFP, CFP and AFP+CFP exceeded 90% (nobody failed AFP test). On the other hand, in terms of Selten's index, AFP+CFP model distinctively performed well. As a comparative experiment, we also collected choice data with smaller feasible sets, where each subject was given 20 feasible sets containing 2 – 5 alternatives out of 10 alternatives. In this case, TRSM achieved the highest value of Selten's index.

From a technical perspective, our revealed preference tests involve combinatorial calculations, which also applies to De Clippel and Rozen (2014)'s test for AFP model.[2] Nevertheless, features of the tests allow us to employ a computing method called *backtracking*, which is an efficient search method in dealing with combinatorial problems.[3] We adopted this method in our testing algorithms, and actually applied them in the simulation and experiment. In that sense, one may regard our simulation and experiment also as the implementation of our algorithm, with which even 10,000 sets of random data can be calculated in acceptable time by using unexceptional computers.

Organization of the paper: In Section 2.1, we introduce limited consideration models that are dealt with in this paper, and in Section 2.2, we briefly review the revealed preference test by De Clippel and Rozen (2014). The theoretical heart of our paper lies in Section 3: we provide a basic idea of our approach in testing limited consideration models, by filling in the details through derivation of our test for AFP model. The revealed preference tests for CFP, AFP+CFP, and rational shortlisting type models are given in Section 4. In Section 5, we deal with issues related to computation and algorithms, including the backtracking method. Finally, respectively in Sections 6.1 and 6.2, we apply our tests to simulation and experimental data.

2 Limited consideration models

2.1 Models

Consider a single-agent decision problem where X is a finite set of alternatives, and $\succ$ is a complete, asymmetric, and transitive preference of an agent, to which we refer as a *strict preference*.[4] If an agent obeys the *rational choice* model, then for every feasible set $A \subset X$, she maximizes her strict preference on A. On the other hand, motivated by evidences contradicting the rational choice theory, a number of alternative decision procedures are proposed in the literature of bounded rationality. Amongst others, in this paper, we focus on *limited*

[2] De Clippel and Rozen (2014) show that their test for AFP model is NP hard.

[3] Classical textbook examples where backtracking is used are the eight queens puzzle, crossword puzzles, and sudoku.

[4] For every $x \in X$, $x \not\succ x$, and for every distinct $x, y \in X$, either $x \succ y$ or $y \succ x$ holds, and for every distinct $x, y, z \in X$, $x \succ y$ and $y \succ z$ imply $x \succ z$.

consideration models, where either consciously or unconsciously, an agent makes a shortlist of alternatives before she chooses an alternative. That is, there exists a *consideration mapping* $\Gamma : 2^X \to 2^X$ such that $\Gamma(A) \subset A$ for every $A \subset X$, and an agent maximizes her strict preference on $\Gamma(A)$, rather than A itself. In what follows, given a consideration mapping Γ, $\Gamma(A)$ is referred to as a *consideration set* on A. Furthermore, in general, we refer to a pair of a consideration mapping and a strict preference $(\succ, \Gamma)$ as a limited consideration model.

In Masatlioglu, Nakajima, and Ozbay (2012), they consider a situation in which an agent cannot recognize all feasible alternatives due to limitation of recognition capacity. There, following psychological literature, a consideration mapping Γ is supposed to have the *attention filter property (AFP)* defined as: for every $A \subset X$ and $x \in A$,

$$x \notin \Gamma(A) \Longrightarrow \Gamma(A \backslash x) = \Gamma(A). \tag{1}$$

In words, the consideration set is not affected when unrecognized elements are removed from a feasible set. Alternatively, (1) is rewritten as: for every $A \subset X$ and $A' \subset A$,

$$\Gamma(A) \subset A \backslash A' \Longrightarrow \Gamma(A \backslash A') = \Gamma(A). \tag{2}$$

In what follows, when Γ obeys AFP, we refer to $(\succ, \Gamma)$ as a limited consideration model with AFP, or simply, an *AFP model*.

As an alternative structure of a consideration mapping, Lleras, Masatlioglu, Nakajima, and Ozbay (2017) consider the following restriction: for every $A' \subset A''$ and $x \in A'$,

$$x \notin \Gamma(A') \Longrightarrow x \notin \Gamma(A''). \tag{3}$$

In words, if an alternative is not recognized in a smaller feasible set, then it cannot be recognized in a larger feasible set. This seems plausible if an agent has limited capacity of recognition. Equivalently, (3) can be written as: for every $A' \subset A''$,

$$\Gamma(A'') \cap A' \subset \Gamma(A'). \tag{4}$$

This condition is equivalent to the monotonicity of the set of unrecognized alternatives. We say that Γ obeys the *competition filter property (CFP)* if it obeys (3), or equivalently (4). A limited consideration model $(\succ, \Gamma)$ is referred to as a *CFP model* when Γ obeys CFP. It is known that this type of restriction on Γ characterizes some conscious shortlisting behavior such as the *order rationalization* model by Cherepanov, Feddersen, and Sandroni (2013) and the *categorize-then-choose* model by Manzini and Mariotti (2012). This property is also used in the *limited consideration with status quo* model in Dean, Kibris, and Masatlioglu (2017).

If we admit that both AFP and CFP are reasonable, then it is natural to consider the joint of AFP and CFP. Indeed, as pointed out in Lleras, Masatlioglu, Nakajima, and Ozbay (2015), both AFP and CFP are plausible in a number of real-world examples. For instance, consider the situations in which an agent pays attention to: (a) n-most advertised commodities; (b) all commodities of a specific brand, and if there are none available, then all commodities of another specific brand; or (c) n-top candidates in each field in job markets. All of these decision procedures derive consideration mappings satisfying both AFP and CFP. A pair $(\succ, \Gamma)$ is referred to as a limited consideration model with AFP+CFP, or an *AFP+CFP model* in short, if Γ obeys AFP+CFP.

Limited consideration models with CFP and those with AFP+CFP can be related to Manzini and Mariotti (2007)'s two-step decision procedure called a *rational shortlist method.* There, an agent has a preference relation for each step, say $\succ'$ and $\succ$, and for every $A \subset X$, an agent firstly makes a shortlist $\Gamma(A)$ such that

$$\Gamma(A) = \{x \in A : \nexists x' \in A \text{ such that } x' \succ' x\}, \tag{5}$$

and then, in the second step, she maximizes her second step preference relation $\succ$ on $\Gamma(A)$. In Manzini and Mariotti (2007), the first step preference $\succ'$ is just assumed to be acyclic, while Au and Kawai (2011) deal with the case where $\succ'$ is asymmetric and transitive.[5] We say that Γ obeys the *(transitive) rational shortlist method*, or in short, RSM (TRSM), if it can be described as (5) by using an acyclic (asymmetric and transitive) binary relation $\succ'$. By abuse of terminology, we refer to $(\succ, \Gamma)$ as an *RSM (TRSM) model*, if Γ obeys RSM (TRSM). If $(\succ, \Gamma)$ is an RSM model and $x \notin \Gamma(A)$ for some $x \in A$, then there exists some $x' \in A$ such that $x' \succ' x$. Then, for every $A' \supset A$, $x \notin \Gamma(A')$, and hence, an RSM model is a special case of a CFP model. When $\succ'$ is asymmetric and transitive, which is a TRSM case, Γ defined in (5) also obeys AFP, i.e., it obeys AFP+CFP. Moreover, one can confirm that $(\succ, \Gamma)$ is a TRSM

[5] In Manzini and Mariotti (2007), they assumed that both $\succ'$ and $\succ$ are just asymmetric. However, since they also assume that the choice function is nonempty for all $A \subset X$, it is clear that $\succ'$ must be acyclic (otherwise $\Gamma(A)$ would be empty for some A).

model if and only if it is an RSM model obeying AFP. We state it as a proposition, since it is new in the literature.

Proposition 1. *A limited consideration model* $(\succ, \Gamma)$ *is a TRSM model, if and only if it is an RSM model obeying AFP.*

Note that, while every RSM (TRSM) model obeys CFP (AFP+CFP), there exists a consideration mapping with CFP (AFP+CFP) that cannot be represented as (5) for any acyclic (asymmetric and transitive) binary relation on X. For example, amongst the examples (a) – (c) referred to in introducing an AFP+CFP model, only (b) is consistent with a TRSM model. It is also clear that if $(\succ, \Gamma)$ is a TRSM model, then it is a special case of all other models referred to in this section.

2.2 De Clippel and Rozen's test for AFP model

In the decision theory literature, all decision models stated in the preceding subsection are introduced with revealed preference based characterizations (see Appendix III for details). These characterizations are based on the properties of a *choice function*, which is a function $f : 2^X \to X$ with $f(A) \in A$ for every $A \subset X$ and $f(A)$ is interpreted as an agent's choice when a feasible set is A. In other words, an econometrician is supposed to observe an agent's choices over *all* logically possible feasible sets. This is reasonable in that these characterizations are mainly for clarifying normative aspects of models rather than testing them.

On the other hand, from the viewpoint of testing models, it is more useful to have revealed preference characterizations based on a data set in which an agent's choices on *some* feasible sets are observable. Formally, we consider a data set in the form of $\mathcal{O} = \{(a^t, A^t)\}_{t\in\mathcal{T}}$, where $\mathcal{T} = \{1, 2, ..., T\}$ is the set of indices of observations, $A^t \subset X$ is the feasible set at observation t, and $a^t \in A^t$ is the chosen alternative at $t \in \mathcal{T}$. Throughout this paper, we assume that $A^s \neq A^t$ for $s \neq t$.[6] In general, deriving a revealed preference characterization based on a general data set is nontrivial even if that based on a choice function is known.[7] Concerning limited consideration models, a revealed preference test based on $\mathcal{O} = \{(a^t, A^t)\}_{t\in\mathcal{T}}$ is firstly analyzed by De Clippel and Rozen (2014) mainly for AFP model. To see how their result builds upon the known choice-function-based characterization, we give a brief review of their

[6] Obviously, a choice function is a data set with $\{A^t\}_{t\in\mathcal{T}} = 2^X$.

[7] Even for the rational choice model, for example, a data set requires to check SARP, while it suffices to check WARP (asymmetry of the direct revealed preference relation) if a choice function is observable.

AFP test, while shedding light on a common issue concerning revealed preference tests for limited consideration models based on a general data set.

When an econometrician has access to a choice function f, AFP model is characterized in Masatlioglu et al. (2012) as follows. Define a binary relation $\succ_{\mathrm{AFP}}$ such that for $x', x'' \in X$,

$$x'' \succ_{\mathrm{AFP}} x', \text{ if } x'' = f(A) \text{ and } x'' \neq f(A \backslash x') \text{ for some } A \subset X. \tag{6}$$

As long as AFP is imposed on Γ, this $\succ_{\mathrm{AFP}}$ reveals "true" preference rankings between alternatives. To see this, suppose that $x'' \succ_{\mathrm{AFP}} x'$ holds, or $x'' = f(A) \neq f(A \backslash x')$. Then, $\Gamma(A) \neq \Gamma(A \backslash x')$ must hold, and AFP implies that $x' \in \Gamma(A)$. Hence, $x'' \succ x'$ must hold. Thus, naturally, for a choice function f to be consistent with some AFP model, $\succ_{\mathrm{AFP}}$ must be acyclic, and more substantially, the other direction is also the case, i.e., the acyclicity of $\succ_{\mathrm{AFP}}$ is necessary and *sufficient* for f to be consistent with an AFP model. That is, under the acyclicity of $\succ_{\mathrm{AFP}}$, there exists AFP model $(\succ, \Gamma)$ such that for every $A \subset X$, $f(A) \succ x$ for every $x \in \Gamma(A) \backslash f(A)$.

De Clippel and Rozen (2014) point out that, while $\succ_{\mathrm{AFP}}$ can be similarly defined even under a general data set $\mathcal{O} = \{(a^t, A^t)\}_{t \in \mathcal{T}}$ and its acyclicity is still necessary, it is no longer sufficient for $\mathcal{O}$ to be consistent with some AFP model. We review their example below, where $\succ_{\mathrm{AFP}}$ is acyclic, but the data set cannot be consistent with any AFP model. That is, there is no AFP model $(\succ, \Gamma)$ such that for every $t \in \mathcal{T}$, $a^t \succ x$ for $x \in \Gamma(A^t) \backslash a^t$.[8]

Example 1. *Let $X = \{x_1, x_2, x_3, x_4, x_5\}$, and consider a data set of six observations as below, where for each $t \in \mathcal{T}$, the chosen alternative is underlined (e.g., $a^1 = x_4$, $a^2 = x_5$,and so on):*

$$A^1 = \{x_1, \underline{x_4}\},\ A^2 = \{x_4, \underline{x_5}\},\ A^3 = \{x_1, x_2, \underline{x_3}\},$$
$$A^4 = \{\underline{x_1}, x_3, x_4\},\ A^5 = \{\underline{x_2}, x_3, x_4\},\ A^6 = \{x_2, \underline{x_4}, x_5\}.$$

Here we can define $\succ_{AFP}$ such that $x_1 \succ_{AFP} x_3$ and $x_4 \succ_{AFP} x_2$, which is obviously acyclic. For example, we have $x_1 \succ_{AFP} x_3$, because $a^4 = x_1 \neq a^1$ and $A^1 = A^4 \backslash x_3$. Given $x_1 \succ_{AFP} x_3$ and $x_4 \succ_{AFP} x_2$, it must hold that $x_1 \succ x_3$ and $x_4 \succ x_2$ for the data set to be consistent with an AFP model. This in turn implies that $x_1 \notin \Gamma(A^3)$ and $x_4 \notin \Gamma(A^5)$. Then, AFP requires that

[8] Note that Example 1 has another important implication. For the data set in the example, in fact, we can find a pair $(\succ, \Gamma)$ so that (i) $a^t \succ x$ for every $t \in \mathcal{T}$ and $x \in \Gamma(A^t) \backslash a^t$, and (ii) Γ obeys AFP on observed feasible sets. The data set is *not* consistent with any AFP model, since it is impossible to find any Γ that obeys AFP *on the entire domain* with satisfying $a^t \succ x$ for every $t \in \mathcal{T}$ and $x \in \Gamma(A^t) \backslash a^t$. This is known as the *extendability problem.*

$\Gamma(A^3) = \Gamma(A^3\backslash x_1) = \Gamma(\{x_2, x_3\}) = \Gamma(A^5\backslash x_4) = \Gamma(A^5)$, *which is impossible, because* $a^3 \neq a^5$.

Notice that in the above example, it holds that $a^3, a^5 \in A^3 \cap A^5$, $a^3 \neq a^5$, and $A^3\backslash A^5 = \{x_1\}$ and $A^5\backslash A^3 = \{x_4\}$ are completely excluded respectively from $\Gamma(A^3)$ and $\Gamma(A^5)$. In fact, this is a source of the inconsistency with AFP model. More generally, if a data set is collected from an agent obeying AFP model $(\succ, \Gamma)$, it must hold that $a^s, a^t \in A^s \cap A^t$ and $a^s \neq a^t \Longrightarrow \exists x \in A^s\backslash A^t : x \in \Gamma(A^s)$ or $\exists x \in A^t\backslash A^s : x \in \Gamma(A^t)$. The RHS in turn implies that either $a^s \succ x$ for some $x \in A^s\backslash A^t$ or $a^t \succ x$ for some $x \in A^t\backslash A^s$. The main contribution of De Clippel and Rozen (2014) is showing that this property essentially characterizes data sets derived from AFP models.

Theorem 0. [De Clippel and Rozen] *A data set* $\mathcal{O} = \{(a^t, A^t)\}_{t\in\mathcal{T}}$ *is consistent with AFP model, if and only if there exists an acyclic binary relation* $\succ^*$ *such that*

$$a^s, a^t \in A^s \cap A^t \text{ and } a^s \neq a^t \Longrightarrow \exists x \in A^s\backslash A^t : a^s \succ^* x \text{ or } \exists x \in A^t\backslash A^s : a^t \succ^* x. \quad (7)$$

By using Figure 1, we can see how Theorem 0 strengthens the condition of the acyclicity of $\succ_{\text{AFP}}$. Consider the pair of observations (a^s, A^s) and (a^t, A^t) in Figure 1(a), where $\succ_{\text{AFP}}$ is not explicitly generated. However, as a matter of fact, either $a^s \succ_{\text{AFP}} y'$ (the red arrow) **or** $a^t \succ_{\text{AFP}} x'$ (the blue arrow) is "hidden" behind these observations. Letting $\bar{A} := A^s \cap A^t = \{a^s, a^t\}$, even if a choice there is unobservable, either a^s or a^t must be chosen from $\bar{A}$. Guessing that a^s is chosen from $\bar{A}$, it means that the choice reversal occurs when x' is removed from A^t. Thus, we have a situation of (6) so that $a^t \succ_{\text{AFP}} x'$ or the blue arrow is implied (Figure 1(c)). Similarly, guessing that a^t is chosen from $\bar{A}$, $a^s \succ_{\text{AFP}} y'$ or the red arrow is implied (Figure 1(b)). Notice that each case corresponds to the RHS of the condition (7). Given this, we can loosely interpret Theorem 0 as follows: it requires that for each pair of observations like Figure 1(a), we can make a guess of a hidden $\succ_{\text{AFP}}$ relation so that the collection of them is acyclic even combined with explicitly defined $\succ_{\text{AFP}}$ relations. In Example 1, (a^3, A^3) and (a^5, A^5) are exactly as Figure 1(a), but any possible conjecture on $\succ_{\text{AFP}}$ (either $x_3 \succ_{\text{AFP}} x_1$ or $x_2 \succ_{\text{AFP}} x_4$) would cause a cycle due to the existence of explicitly generated $\succ_{\text{AFP}}$ relations of $x_1 \succ_{\text{AFP}} x_3$ and $x_4 \succ_{\text{AFP}} x_2$.

As seen from the above argument, in dealing a general data set, we need to make a guess of hidden preference relations and check if there exists such a guess that is consistent with the theoretical hypothesis in issue (AFP, in the above case). Although our approach for revealed

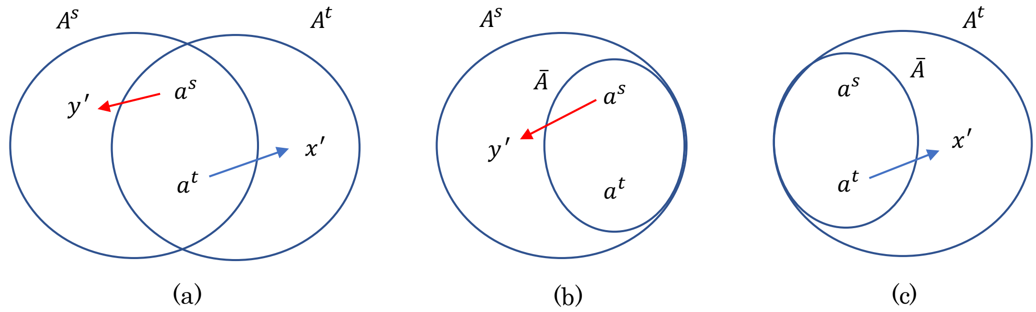

Figure 1: Making a guess of a hidden $\succ_{\text{AFP}}$ relation.

preference tests is distinct from that of De Clippel and Rozen (2014) even for the case of AFP model, it does share this principle.

3 Building up our idea via AFP test

In this section, we put forward a general idea for our revealed preference tests, while clarifying the details through the case of AFP model. Similar to De Clippel and Rozen (2014), we consider a data set in the form of $\mathcal{O} = \{(a^t, A^t)\}_{t \in \mathcal{T}}$. Recall that $\mathcal{T} = \{1, 2, ..., T\}$ is the set of indices of observations, $A^t \subset X$ is the feasible set at observation t, $a^t \in A^t$ is the chosen alternative at $t \in \mathcal{T}$, and $A^s \neq A^t$ is assumed for every $s \neq t$. Note that, concerning notation for a binary relation, we use both $(x, y) \in \mathcal{R}$ and $x\mathcal{R}y$ to represent that x and y are ordered by some $\mathcal{R}$.

Definition 1. A data set $\mathcal{O} = \{(a^t, A^t)\}_{t \in \mathcal{T}}$ is *rationalizable* by a limited consideration model $(\succ, \Gamma)$ if for every $t \in \mathcal{T}$, $a^t \succ x$ whenever $x \in \Gamma(A^t) \backslash a^t$. If $\mathcal{O}$ is rationalizable by $(\succ, \Gamma)$ where $\Gamma : 2^X \to 2^X$ obeys the property $\mathbf{M} \in \{$AFP, CFP, AFP+CFP, RSM, TRSM$\}$ on 2^X, then we say that $\mathcal{O}$ is rationalizable by a limited consideration model $\mathbf{M}$.[9]

Given a data set $\mathcal{O} = \{(a^t, A^t)\}_{t \in \mathcal{T}}$, define the *direct revealed preference* relation $\succ^R$ such that $x'' \succ^R x'$, if $x'' = a^t$ for some $t \in \mathcal{T}$, $x'' \neq x'$, and $x' \in A^t$. It is well known that a data set is consistent with the rational choice model, if and only if $\succ^R$ is acyclic, or the *strong axiom of revealed preference (SARP)* is satisfied. Put otherwise, if a data set $\mathcal{O}$ obeys SARP, then we can find a strict preference $\succ$ such that $(\succ, \Gamma)$ rationalizes $\mathcal{O}$ with Γ being

[9] We require Γ to satisfy the corresponding property on entire domain 2^X rather than the set of observed feasible sets. Hence, our revealed preference tests take into account the extendability problem referred to in footnote 8. Naturally, one could consider the rationalizability by imposing each property only on observed feasible sets, which is done by Tyson (2013) for AFP model.

the identity mapping. It is easy to check that the rational choice model is a special case of AFP, CFP, AFP+CFP, RSM, and TRSM respectively: if Γ is the identity mapping, it obeys all these properties. Hence, our revealed preference tests become substantial when $\mathcal{O}$ contains revealed preference cycles, which is formally defined as a set of pairs $C = \{(x^k, x^{k+1})\}_{k=1}^K$ with $x^k \succ^R x^{k+1}$ for every $k = 1, 2, \ldots, K$, and $x^1 = x^{K+1}$. We refer to each (x^k, x^{k+1}) as an *edge* of this cycle.

Following the definition, in testing if $\mathcal{O} = \{(a^t, A^t)\}_{t \in \mathcal{T}}$ is rationalizable by **M**-model, we need to find *both* $\succ$ and Γ obeying **M**. Given that our revealed preference analysis puts emphasis on testing models from actual data, reducing it to a tractable problem is one of the central issues. Specifically, we can reduce the rationalizablity of a data set to the solvability of a specific constraint satisfaction problem defined on the set of revealed preference cycles. Suppose that a data set contains Q revealed preference cycles in total, and for every $q = 1, 2, ..., Q$, let C_q be the q-th revealed preference cycle. Picking up one edge for each q-th cycle, we can construct a sequence $(c_1, c_2, ..., c_Q) \in \times_{q=1}^{Q} C_q$. In particular, we say that a sequence $(c_1, c_2, ..., c_Q)$ is a *traverse* across cycles, if the set $\{c_q\}_{q=1}^{Q}$ is acyclic as a binary relation on X.[10] (Note that each c_q is an ordered pair of elements in X, and hence $\{c_q\}_{q=1}^{Q}$ can be regarded as a binary relation on X.) For each $\mathbf{M} \in$ {AFP, CFP, AFP+CFP, RSM, TRSM}, we identify a constraint, which we call **M**-condition, so that the existence of a traverse obeying it is equivalent to the rationalizability of a data set by **M**-model. While searching such a traverse involves combinatorial calculations, as explained in Section 5, every **M**-condition has a nice structure that enables us to apply a simple but powerful search algorithm called *backtracking*.

In the rest of this section, we first derive a common starting point of all our revealed preference tests. Then, we proceed to the case of AFP model to see how a model-specific constraint on a traverse, which actually works as a test, is identified. The essence of our testing procedure is common across models, though other models are postponed to Section 4.

[10] The motivation of the term "traverse" is due to the way it "cuts through" cycles (every cycle has an edge in it) in a "non-circular" fashion (it is acyclic).

A common starting point of our revealed preference tests

Suppose that a data set $\mathcal{O}$ is generated by some limited consideration model $(\succ, \Gamma)$. Assuming that there are Q revealed preference cycles, define $S_\succ$ such that

$$S_\succ = \left\{ (x, y) \in \bigcup_{q=1}^{Q} C_q : y \succ x \right\}, \tag{8}$$

which is interpreted as the set of "false" revealed preference relations in cycles (in that $x \succ^R y$ but $y \succ x$). Since $\succ$ is a strict preference, $S_\succ$ is acyclic as a binary relation, and every revealed preference cycle must contain at least one element of $S_\succ$, i.e., $S_\succ \cap C_q \neq \varnothing$ for every $q = 1, 2, ..., Q$. Consider a traverse $c = (c_1, c_2, ..., c_Q) \in \times_{q=1}^{Q} C_q$ of which each c_q is chosen from $S_\succ \cap C_q$, and let $S_c = \{c_q\}_{q=1}^{Q}$. Note that such a profile c is indeed a traverse, since $S_c \subset S_\succ$ and $S_\succ$ is acyclic. Defining S_c^{-1} as the inverse relation of S_c, or $(y, x) \in S_c^{-1} \iff (x, y) \in S_c$, it is actually a part of the preference relation: $(y, x) \in S_c^{-1} \implies (x, y) \in S_c \subset S_\succ$, and it is clear that $(x, y) \in S_\succ \implies (y, x) \in \succ$. Hence, letting

$$B_c^t = \{y \in A^t : y S_c^{-1} a^t\} \tag{9}$$

for every $t \in \mathcal{T}$, it is a set of feasible alternatives better than the chosen alternative a^t. Obviously, such a set must be excluded from $\Gamma(A^t)$, or for every $t \in \mathcal{T}$, $\Gamma(A^t) \subset A^t \backslash B_c^t$ must hold.

Summarizing, if a data set is generated from a limited consideration model $(\succ, \Gamma)$, then there exists a traverse across revealed preference cycles $c = (c_1, c_2, ..., c_Q)$ such that $\Gamma(A^t) \subset A^t \backslash B_c^t$ for every $t \in \mathcal{T}$. We record this simple observation for future references, as it is a common starting point of all our revealed preference tests.

Fact 1. *Suppose that a data set $\mathcal{O} = \{(a^t, A^t)\}_{t \in \mathcal{T}}$ is collected from an agent obeying some limited consideration model $(\succ, \Gamma)$. Then, there exists a traverse $c = (c_1, c_2, ..., c_Q) \in \times_{q=1}^{Q} C_q$ such that $\Gamma(A^t) \subset A^t \backslash B_c^t$ for every $t \in \mathcal{T}$, where $B_c^t = \{y \in A^t : y S_c^{-1} a^t\}$.*

When a specific restriction **M** is imposed on Γ, we can derive a stronger constraint on a traverse, which we call **M**-condition for each **M** $\in$ {AFP, CFP, AFP+CFP, RSM, TRSM}. Every **M**-condition is constructed based on Fact 1 and the shape restriction on Γ specific to each model, and the testing procedure for each model is reduced to searching for a traverse obeying **M**-condition.

Note that, actually, in checking the existence of a traverse obeying **M**-condition, it suffices to search within the set of *essential* revealed preference cycles. Formally, a revealed preference cycle is said to be essential, if it does not contain any cycles except for itself. For example, a cycle like $x \succ^R y \succ^R x \succ^R z \succ^R x$ is *not* essential, since it contains two (essential) cycles $x \succ^R y \succ^R x$ and $x \succ^R z \succ^R x$. In what follows, we only consider essential revealed preference cycles, i.e., when we say there are Q cycles in total, it means that there are Q essential revealed preference cycles. The validity of this reduction is proved in Section 5.1 as a part of computational issue.

Testing AFP model based on Fact 1

In the rest of this section, we consider the case of $\mathbf{M}$ = AFP. Consider a data set $\mathcal{O} = \{(a^t, A^t)\}_{t\in\mathcal{T}}$ derived from an agent obeying an AFP model $(\succ, \Gamma)$, and suppose that there are Q revealed preference cycles. Then, by Fact 1, there exists a traverse across cycles $c = (c_1, c_2, ..., c_Q)$ such that for every $t \in \mathcal{T}$, $\Gamma(A^t) \subset A^t \backslash B^t_c$, and AFP casts further restrictions. If both $A^t \backslash B^t_c \subset (A^s \cap A^t)$ and $A^s \backslash B^s_c \subset (A^s \cap A^t)$ hold for some $s, t \in \mathcal{T}$, it implies that both $\Gamma(A^t) \subset (A^s \cap A^t) \subset A^t$ and $\Gamma(A^s) \subset (A^s \cap A^t) \subset A^s$ hold. Then, by AFP, it must be the case that $\Gamma(A^s) = \Gamma(A^s \cap A^t) = \Gamma(A^t)$, which in turn implies that $a^s = a^t$. Thus, when a data set is rationalizable by an AFP model, there must exist a traverse obeying the following condition, and more substantially, the other direction is also the case.

AFP-condition: Given a data set $\mathcal{O} = \{(a^t, A^t)\}_{t\in\mathcal{T}}$ containing Q revealed preference cycles, a traverse $c = (c_1, c_2, ..., c_Q) \in \times_{q=1}^{Q} C_q$ obeys AFP-condition, if for every $s, t \in \mathcal{T}$,

$$\left[(A^s \backslash B^s_c) \cup (A^t \backslash B^t_c)\right] \subset (A^s \cap A^t) \Longrightarrow a^s = a^t. \tag{10}$$

Theorem 1. *A data set $\mathcal{O} = \{(a^t, A^t)\}_{t\in\mathcal{T}}$ is rationalizable by an AFP model, if and only if there exists a traverse obeying AFP-condition.*

By Theorem 1, testing AFP model is reduced to checking the existence of a traverse obeying AFP-condition, which is obviously a constraint satisfaction problem. Recall that, if a traverse $c = (c_1, c_2, ..., c_Q)$ is constructed from a preference relation $\succ$ such that $c_q \in S_\succ \cap C_q$ for every q, the corresponding S^{-1}_c is a part of $\succ$. Thus, when a true preference $\succ$ is unknown, a traverse c, or the corresponding S^{-1}_c can be interpreted as a partial guess of a hidden preference relation. In this sense, our test is searching for a guess of preference obeying a restriction derived from the model in issue, which is reminiscent of Theorem 0. Then, by (9), B_c^t is interpreted as a guess of the set of alternatives better than a choice. As we refer to in Fact 1, such a set must be excluded from $\Gamma(A^t)$, and since AFP is a shape restriction on Γ, it naturally casts some restriction on "better-than sets" $\{B_c^t\}_{t\in\mathcal{T}}$, which is nothing but AFP-condition. The above aspect is also shared by other **M**-conditions and tests based on them in Section 4.

The formal proof of Theorem 1 is postponed to Appendix I, but we here provide a basic idea for proving the substantial direction, or "if" part. Our proof consists the following three steps:

(i) Fix a traverse c obeying AFP-condition and define $\Gamma : 2^X \to 2^X$ such that

$$\begin{aligned}\Gamma(A) &= A\backslash B_c^t, \text{ if } A^t\backslash B_c^t \subset A \subset A^t \\ &= A, \text{ otherwise,}\end{aligned} \tag{11}$$

which is actually well-defined: based on a traverse obeying AFP-condition, if $A^t\backslash B_c^t \subset A \subset A^t$ and $A^s\backslash B_c^s \subset A \subset A^s$ hold for some $s \neq t$, then we have $A\backslash B_c^t = A\backslash B_c^s$. In addition, Γ actually obeys AFP.

(ii) Based on Γ constructed in the previous step, we define a binary relation $\succ^*$ such that $x'' \succ^* x'$, if for some $t \in \mathcal{T}, x'' = a^t, x' \in \Gamma(A^t)$ and $x'' \neq x'$, and show that it is acyclic.

(iii) Letting $\succ$ be a linear extension of $\succ^*$, we confirm that $(\succ, \Gamma)$ rationalizes the data set.

In fact, all our revealed preference tests can be proved through the above three-step procedure, though the construction of Γ differs across models in issue.

Remark: We here look at AFP-condition from a slightly different viewpoint. Suppose that a data set $\mathcal{O} = \{(a^t, A^t)\}_{t\in\mathcal{T}}$ is collected from an AFP model $(\succ, \Gamma)$, and define for each t, the set of alternatives better than a^t as $B_\succ^t = \{y \in A^t : y \succ a^t\}$. Then, one can confirm that (10) is satisfied even if B_c^t is replaced with $B_\succ^t$. In fact, this modified version of AFP-condition characterizes the set of preferences that can rationalize $\mathcal{O}$ combined with some Γ obeying AFP.[11] Put otherwise, given a data set $\mathcal{O}$ and the hypothesis of Γ obeying AFP, the set of possible preferences of an agent is equal to the set of preferences obeying AFP-condition. In fact, this property is also shared by other **M**-conditions.

To see how Theorem 1 works as a test, we provide the following two numerical examples.

[11] The necessity is almost obvious. Given a preference $\succ$ of which $\{B_\succ^t\}$ obeys (10), then $\succ$ can rationalize the data, combined with Γ defined as (11) replacing B_c^t with $B_\succ^t$.

The first example is the same with the one in Section 2.2, which is not rationalized by any AFP model, and we confirm how our test rejects this data set. On the other hand, in the second example, there is a traverse obeying AFP model and the data set is rationalized by an AFP model.

Example 1 (continued)**.** *Consider the data set in Example 1. In this data set, there are seven essential revealed preference cycles.*[12] *We show that it is impossible to find a traverse c obeying AFP-condition, by focusing on cycles* $x_1 \succ^R x_3 \succ^R x_1$ *and* $x_2 \succ^R x_4 \succ^R x_2$*. Recall that by the definition of a traverse, every cycle must have at least one edge in c. Consider any traverse c for which edge* (x_1, x_3) *is selected at* $x_1 \succ^R x_3 \succ^R x_1$*. This yields* $(x_1, x_3) \in S_c$*, which in turn implies* $x_3 \in B_c^4$*, since* $x_3 \in A^4$ *and* $(x_3, x_1) \in S_c^{-1}$ *with* $x_1 = a^4$*. Then, regardless of* B_c^1*, we have* $\left[(A^1 \backslash B_c^1) \cup (A^4 \backslash B_c^4)\right] \subset \{x_1, x_4\} = (A^1 \cap A^4)$*, but* $x_4 = a^1 \neq a^4 = x_1$*, a violation of AFP-condition. Similarly, any traverse c in which edge* (x_4, x_2) *is selected from* $x_2 \succ^R x_4 \succ^R x_2$ *leads to a violation of (10) at observations 2 and 6. Finally, we consider any traverse c with* (x_3, x_1) *and* (x_2, x_4) *being selected respectively from* $x_1 \succ^R x_3 \succ^R x_1$ *and* $x_2 \succ^R x_4 \succ^R x_2$*. This implies that* $x_1 \in B_c^3$ *and* $x_4 \in B_c^5$*, and hence,* $\left[(A^3 \backslash B_c^3) \cup (A^5 \backslash B_c^5)\right] \subset \{x_2, x_3\} = A^3 \cap A^5$*. However,* $x_3 = a^3 \neq a^5 = x_2$*, which is a violation of (10). As a result, we cannot find any traverse obeying (10), or equivalently, the data set in question is not rationalizable by AFP model.*

Example 2. *Let* $X = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7\}$*, and consider a data set of five observations as follows, where for each* $t \in \mathcal{T}$*, the chosen alternative is underlined:*

$$A^1 = \{\underline{x_1}, x_2, x_3\},\ A^2 = \{x_1, \underline{x_2}, x_4, x_6\},\ A^3 = \{x_1, \underline{x_3}, x_5, x_7\},$$
$$A^4 = \{x_2, \underline{x_4}, x_6\},\ A^5 = \{x_3, \underline{x_5}, x_7\}.$$

There are four essential revealed preference cycles: $C_1 : x_1 \succ^R x_2 \succ^R x_1$*,* $C_2 : x_1 \succ^R x_3 \succ^R x_1$*,* $C_3 : x_2 \succ^R x_4 \succ^R x_2$*, and* $C_4 : x_3 \succ^R x_5 \succ^R x_3$*; and hence there must exist at least one "false"* $\succ^R$*-ordered pair in each of them. One possible traverse is* $c = ((x_1, x_2), (x_1, x_3), (x_2, x_4), (x_3, x_5))$*. Fixing such c, we have the corresponding binary relation* $S_c = \{(x_1, x_2), (x_1, x_3), (x_2, x_4), (x_3, x_5)\}$*, and then* $\{B_c^t\}_{t \in \mathcal{T}}$ *is derived as follows. Since we have* $x_1 = a^1$*,* $x_2, x_3 \in A^1$*, and* $(x_2, x_1), (x_3, x_1) \in S_c^{-1}$*, it follows from (9) that* $B_c^1 = \{x_2, x_3\}$*.*

[12] The essential cycles are: (1) $x_1 \succ^R x_3 \succ^R x_1$; (2) $x_1 \succ^R x_4 \succ^R x_1$; (3) $x_2 \succ^R x_3 \succ^R x_2$; (4) $x_2 \succ^R x_4 \succ^R x_2$; (5) $x_4 \succ^R x_5 \succ^R x_4$; (6) $x_1 \succ^R x_3 \succ^R x_2 \succ^R x_4 \succ^R x_1$; and (7) $x_1 \succ^R x_4 \succ^R x_2 \succ^R x_3 \succ^R x_1$.

t	1	2	3	4	5
B^t	$\{x_2, x_3\}$	$\{x_4\}$	$\{x_5\}$	$\varnothing$	$\varnothing$
$A^t \backslash B^t$	$\{x_1\}$	$\{x_1, x_2, x_6\}$	$\{x_1, x_3, x_7\}$	$\{x_2, x_4, x_6\}$	$\{x_3, x_5, x_7\}$

Table 1: Sets $\{B_c^t\}_{t \in \mathcal{T}}$ corresponding to traverse c in Example 2.

Similarly, we have $x_2 = a^2$, $x_4 \in A^2$, and $(x_4, x_2) \in S_c^{-1}$, so $B_c^2 = \{x_4\}$. Following an analogous procedure, we have $B_c^3 = \{x_5\}$, and $B_c^4 = B_c^5 = \varnothing$. Now we show that the traverse $c = ((x_1, x_2), (x_1, x_3), (x_2, x_4), (x_3, x_5))$ actually succeeds in satisfying AFP-condition. Indeed, as we can see from Table 1, there is no pair $s, t \in \mathcal{T}$ such that $\left[(A^s \backslash B_c^s) \cup (A^t \backslash B_c^t)\right] \subset (A^s \cap A^t)$, and AFP-condition is satisfied for this traverse. Thus, this data set is rationalizable by an AFP model.

We conclude this section with the reference to the connection with Theorem 0 by De Clippel and Rozen (2014). Since Theorems 0 and 1 characterize the same model, the existence of a traverse obeying AFP-condition must be equivalent to the existence of a binary relation $\succ^*$ obeying (7) in Theorem 0. We provide a direct proof for this equivalence as follows.

If there exists a traverse $c = (c_1, c_2, ..., c_Q)$ obeying AFP-condition, let $\succ^*$ be the one defined in step (ii) of the proof, i.e., $x'' \succ^* x'$ if $x'' = a^t$ for some $t \in \mathcal{T}$ and $x' \in \Gamma(A^t)$, with Γ being defined by (11). The acyclicity of this binary relation will be proved in Appendix I, and here we show that it obeys (7). Suppose not: for some $s, t \in \mathcal{T}$, it holds that $a^s, a^t \in (A^s \cap A^t)$ with $a^s \neq a^t$, $a^t \not\succ^* x$ for all $x \in A^t \backslash A^s$, and $a^s \not\succ^* x$ for all $x \in A^s \backslash A^t$. Since Γ is defined by (11), this implies that $(A^t \backslash A^s) \subset B_c^t$ and $(A^s \backslash A^t) \subset B_c^s$. This in turn implies that $\left[(A^s \backslash B_c^s) \cup (A^t \backslash B_c^t)\right] \subset (A^s \cap A^t)$, but since we have $a^s \neq a^t$, this means the violation of AFP-condition.

To see the other direction, let $\succ^*$ be an acylic binary relation obeying (7), and let $\succ$ be a linear extension of $\succ^*$. Define $S_\succ$ as in (8), and consider a traverse $c = (c_1, c_2, ..., c_Q)$ of which c_q is taken from $S_\succ \cap C_q$ for every $q = 1, 2, ..., Q$. We claim that this traverse obeys AFP-condition. Suppose to the contrary that, for some $s, t \in \mathcal{T}$, we have $\left[(A^s \backslash B_c^s) \cup \left(A^t \backslash B_c^t\right)\right] \subset (A^s \cap A^t)$ and $a^s \neq a^t$. Since $S_c^{-1} \subset \succ$, letting $B_\succ^t = \{y \in A^t : y \succ a^t\}$, it holds that $B_c^t \subset B_\succ^t$ for every $t \in \mathcal{T}$, which in turn implies that $\left[(A^s \backslash B_\succ^s) \cup \left(A^t \backslash B_\succ^t\right)\right] \subset \left[(A^s \backslash B_c^s) \cup \left(A^t \backslash B_c^t\right)\right] \subset (A^s \cap A^t)$. Then, there is no $x \in A^s \backslash A^t$ with $a^s \succ x$, since $(A^s \backslash B_\succ^s) \subset A^t$ holds. Similarly, we cannot find any $x \in A^t \backslash A^s$ with $a^t \succ x$. As $\succ^* \subset \succ$ holds, this in turn implies that $\succ^*$ cannot satisfy (7), which is a contradiction.

4 Testing CFP, AFP+CFP, and rational shortlisting

In this section, we provide revealed preference tests for CFP, AFP+CFP (Section 4.1), RSM, and TRSM models (Section 4.2). One could confirm that all these tests are built upon a general idea explained in the preceding section, and the structure of tests are parallel to that of the test for AFP model. As stated in Section 3, any of limited consideration models in this paper cannot be refuted, if a data set does not contain revealed preference cycles. In Section 4.3, we strengthen this conclusion: all our revealed preference tests do not "bite" unless there exists some $s, t \in \mathcal{T}$ such that $a^s \succ^R a^t \succ^R a^s$, or the *weak axiom of revealed preference (WARP)* is violated. Put otherwise, not only cyclical choices, but an explicit choice reversal between some pair of alternatives must be observed for limited consideration models in this paper to be refutable.

4.1 CFP-condition and AFP+CFP-condition

Suppose that $\mathcal{O} = \{(a^t, A^t)\}_{t \in \mathcal{T}}$ is collected from an agent obeying a CFP model $(\succ, \Gamma)$, and let Q be the number of revealed preference cycles in the data. By Fact 1, there exists a traverse $c = (c_1, c_2, \ldots, c_Q) \in \times_{q=1}^{Q} C_q$ such that $\Gamma(A^t) \subset A^t \backslash B_c^t$ for every $t \in \mathcal{T}$. Bearing this in mind, consider any $s, t \in \mathcal{T}$ such that $A^s \subset A^t$. Then, considering CFP defined in (4), it must hold that $\Gamma(A^t) \cap A^s \subset \Gamma(A^s)$. In addition, since $\Gamma(A^s) \subset A^s \backslash B_c^s$, this implies that $\Gamma(A^t) \cap B_c^s = \varnothing$, which in turn implies that $a^t \notin B_c^s$. In fact, this simple observation completely characterizes whether a data set is consistent with a CFP model.

CFP-condition: Given a data set $\mathcal{O} = \{(a^t, A^t)\}_{t \in \mathcal{T}}$ containing Q revealed preference cycles, a traverse $c = (c_1, c_2, \ldots, c_Q) \in \times_{q=1}^{Q} C_q$ obeys CFP-condition, if for every $s, t \in \mathcal{T}$,

$$A^s \subset A^t \Longrightarrow a^t \notin B_c^s. \tag{12}$$

Theorem 2. *A data set $\mathcal{O} = \{(a^t, A^t)\}_{t \in \mathcal{T}}$ is rationalizable by a limited consideration model with CFP, if and only if there exists a traverse obeying CFP-condition.*

Remark: As stated in Section 2, it is known in the literature that CFP model is equivalent to the categorize-then-choose model by Manzini and Mariotti (2012) and the order rationalization model by Cherepanov, Feddersen, and Sandroni (2013). Thus when we want to test whether a data set is consistent with these models, it suffices to apply Theorem 2.

Then, we proceed to the case of AFP+CFP models. Clearly, if a data set $\mathcal{O} = \{(a^t, A^t)\}_{t\in\mathcal{T}}$ is rationalizable by an AFP+CFP model, then it is also consistent with both AFP model and CFP model. Hence, by Theorems 1 and 2, such a data set must have a traverse obeying both AFP-condition and CFP-condition. However, as we shall show in the example at the end of this subsection, the joint of AFP-condition and CFP-condition is insufficient to characterize the observable restrictions of AFP+CFP model.

To clarify a necessary condition, let $\mathcal{O} = \{(a^t, A^t)\}_{t\in\mathcal{T}}$ be a data set collected from an agent obeying an AFP+CFP model $(\succ, \Gamma)$, which is again assumed to have Q revealed preference cycles. Similar to the previous cases, by Fact 1, there exists a traverse $c = (c_1, c_2, \ldots, c_Q)$ such that the corresponding sets $\{B_c^t\}_{t\in\mathcal{T}}$ obey $\Gamma(A^t) \subset A^t \backslash B_c^t$ for every $t \in \mathcal{T}$. By using both AFP and CFP, it can be extended such that for every $s, t \in \mathcal{T}$,

$$(A^s \backslash B_c^s) \subset A^t \Longrightarrow \Gamma(A^t) \subset A^t \backslash B_c^s. \tag{13}$$

This can be shown as follows. Note that when $(A^s \backslash B_c^s) \subset A^t$ holds, we have $\Gamma(A^s) \subset A^s \backslash B_c^s \subset (A^s \cap A^t) \subset A^s$. Then, by AFP, $\Gamma(A^s) = \Gamma(A^s \cap A^t)$ must hold, which in turn implies that $x \in B_c^s \Longrightarrow x \notin \Gamma(A^s \cap A^t)$. In addition, since $(A^s \cap A^t) \subset A^t$, CFP implies that $x \in B_c^s \Longrightarrow x \notin \Gamma(A^t)$, which is nothing but (13). For $a^t \in \Gamma(A^t)$ to hold, we must have

$$(A^s \backslash B_c^s) \subset A^t \Longrightarrow a^t \notin B_c^s. \tag{14}$$

Now we turn to extending (13) and (14), and show the following: for every $r, s, t \in \mathcal{T}$,

$$[(A^r \cup A^s) \backslash (B_c^r \cup B_c^s)] \subset A^t \Longrightarrow \Gamma(A^t) \subset A^t \backslash (B_c^r \cup B_c^s). \tag{15}$$

Recall that under the traverse c at hand, both $\Gamma(A^r) \subset A^r \backslash B_c^r$ and $\Gamma(A^s) \subset A^s \backslash B_c^s$ hold. Since Γ obeys CFP, it holds that

$$x \in B_c^r \Longrightarrow x \notin \Gamma(A^r \cup A^s) \text{ and } x \in B_c^s \Longrightarrow x \notin \Gamma(A^r \cup A^s),$$

which implies $\Gamma(A^r \cup A^s) \subset [(A^r \cup A^s) \backslash (B_c^r \cup B_c^s)]$. Since $[(A^r \cup A^s) \backslash (B_c^r \cup B_c^s)] \subset A^t$ is assumed, we have

$$[(A^r \cup A^s) \backslash (B_c^r \cup B_c^s)] \subset [A^t \cap (A^r \cup A^s)] \subset (A^r \cup A^s),$$

and AFP implies that $\Gamma(A^t \cap (A^r \cup A^s)) = \Gamma(A^r \cup A^s) \subset [(A^r \cup A^s)\backslash(B^r \cup B^s)]$. Finally, combining $[A^t \cap (A^r \cup A^s)] \subset A^t$ and CFP, we have $x \in (B^r_c \cup B^s_c) \Longrightarrow x \notin \Gamma(A^t)$ as desired. Gathered together with the fact that $a^t \in \Gamma(A^t)$, the condition (15) in turn implies that for every $r, s, t \in \mathcal{T}$,

$$[(A^r \cup A^s)\backslash(B^r_c \cup B^s_c)] \subset A^t \Longrightarrow a^t \notin (B^r_c \cup B^s_c). \tag{16}$$

By an inductive argument, we can extend (15) and (16) for any subset of indices $\tau \subset \mathcal{T}$ such that $\left(\bigcup_{r\in\tau} A^r \backslash \bigcup_{r\in\tau} B^r_c\right) \subset A^t$. Namely, by the extension of (16), a data set collected from an agent obeying AFP+CFP must have a traverse across revealed preference cycles obeying the following condition.

AFP+CFP-condition: Given a data set $\mathcal{O} = \{(a^t, A^t)\}_{t\in\mathcal{T}}$ containing Q revealed preference cycles, a traverse $c = (c_1, c_2, \ldots, c_Q) \in \times^Q_{q=1} C_q$ obeys AFP+CFP-condition, if for every $t \in \mathcal{T}$ and any set of indices $\tau \subset \mathcal{T}$,

$$\left(\bigcup_{r\in\tau} A^r \Big\backslash \bigcup_{r\in\tau} B^r_c\right) \subset A^t \Longrightarrow a^t \notin \bigcup_{r\in\tau} B^r_c. \tag{17}$$

While it looks less obvious than the cases of AFP and CFP, the existence of a traverse obeying the above does characterize the rationalizability by AFP+CFP model.

Theorem 3. *A data set $\mathcal{O} = \{(a^t, A^t)\}_{t\in\mathcal{T}}$ is rationalizable by an AFP+CFP model, if and only if there exists a traverse obeying AFP+CFP-condition.*

Finally, we point out that the joint of AFP-condition and CFP-condition does not work as a test for rationalizability by an AFP+CFP model. In the example below, a data set contains a traverse obeying both AFP-condition and CFP-condition, and hence it is rationalizable respectively by an AFP model and a CFP model. However, it does *not* contain any traverse obeying AFP+CFP-condition, or equivalently, it is not rationalizable by any AFP+CFP model.

Example 2 (continued)**.** *Consider the data set given in Example 2. It is already shown that traverse $c = ((x_1, x_2), (x_1, x_3), (x_2, x_4), (x_3, x_5))$ succeeds in satisfying AFP-condition. Here, we start from showing that it also succeeds with CFP-condition. Recall that the binary relation corresponding to this traverse is $S_c = \{(x_1, x_2), (x_1, x_3), (x_2, x_4), (x_3, x_5)\}$, and the relevant sets $\{B^t_c\}_{t\in\mathcal{T}}$ are summarized in Table 1. Looking at the data set and Table 1, one can confirm that CFP-condition is satisfied: $A^4 \subset A^2$ and $A^5 \subset A^3$, but $a^2 = x_2 \notin B^4$ and $a^3 = x_3 \notin B^5$.*

Thus, $c = ((x_1, x_2), (x_1, x_3), (x_2, x_4), (x_3, x_5))$ succeeds both in AFP-condition and CFP-condition. However, this traverse violates AFP+CFP-condition. Indeed, since $\{x_1\} = A^1 \backslash B_c^1 \subset A^2$ and $x_2 = a^2 \in B_c^1$, (14) is violated, let alone AFP+CFP-condition. In addition, as shown below, this is the only traverse that obeys both AFP-condition and CFP-condition (thus, there is no traverse that can satisfy AFP+CFP-condition). For a traverse to satisfy CFP-condition, the traverse (or the corresponding S_c) can contain neither (x_4, x_2) nor (x_5, x_3). To see this, suppose that $(x_4, x_2) \in S_c$. Then we have $x_2 \in B_c^4$, $A^4 \subset A^2$, and $a^2 = x_2 \in B_c^4$, which violates CFP-condition. Having $(x_5, x_3) \in S_c$ leads to a similar violation of CFP-condition. Therefore, from the third and fourth cycles, it must be the case that $c_3 = (x_2, x_4)$ and $c_4 = (x_3, x_5)$. Furthermore, for a traverse c to satisfy AFP-condition, $(x_2, x_1) \notin S_c$ and $(x_3, x_1) \notin S_c$ must hold whenever $(x_2, x_4), (x_3, x_5) \in S_c$. To see this, consider traverse $c' = ((x_2, x_1), (x_1, x_3), (x_2, x_4), (x_3, x_5))$. Then we have $B_c^2 = \{x_1, x_4\}$, and thus

$$\{x_2, x_6\} = A^2 \backslash B_c^2 \subset A^4 \subset A^2 = \{x_1, x_2, x_4, x_6\},$$

but $x_2 = a^2 \neq a^4 = x_4$, which is a violation of AFP-condition. The cases of traverse $c'' = ((x_1, x_2), (x_3, x_1), (x_2, x_4), (x_3, x_5))$ and $c''' = ((x_2, x_1), (x_3, x_1), (x_2, x_4), (x_3, x_5))$ respectively lead to similar violations of AFP-condition.

4.2 RSM/TRSM-condition

If a data set $\mathcal{O} = \{(a^t, A^t)\}_{t \in \mathcal{T}}$ is collected from an agent obeying a (transitive) rational shortlist method $(\succ, \Gamma)$, then it is consistent with a CFP model (AFP+CFP model), as stated in Section 2.1. However, it is not difficult to find a data set that is consistent with CFP-condition (AFP+CFP-condition), but inconsistent with any RSM (TRSM) model. Indeed, for a data set to be rationalizable by an RSM model, it must hold that for every $r, s, t \in \mathcal{T}$ with $A^r = A^s \cup A^t$,

$$a^s = a^t \Longrightarrow a^r = a^s = a^t, \tag{18}$$

which is not guaranteed by the existence of a traverse obeying CFP-condition/AFP+CFP-condition.[13] In this subsection, we provide a test for RSM (TRSM) models.

[13]If an agent obeys a rational shortlist method, $\Gamma(A^r) \subset \Gamma(A^s) \cup \Gamma(A^t)$ is obvious. In addition, $x = a^t = a^s$ implies that no element in $A^s \cup A^t = A^r$ can dominate x with respect to the first step preference, and x dominates with

Suppose that an agent has two preferences $\succ'$ and $\succ$, where the former is merely acyclic while the latter is a strict preference, and that a consideration mapping Γ is defined as (5). Similar to the previous models, suppose that a data set is collected from such an agent, and that it contains Q revealed preference cycles. Then, Fact 1 implies that there exists a traverse $c = (c_1, c_2, \ldots, c_Q)$ such that the corresponding $\{B_c^t\}_{t\in\mathcal{T}}$ satisfies $\Gamma(A^t) \subset A^t \backslash B_c^t$ for every $t \in \mathcal{T}$. Since an agent obeys an RSM (TRSM) model, for every $x' \in B_c^t$, there exists some $x'' \in A^t \backslash x'$ such that $x'' \succ' x'$. This in turn implies that x' is not considered as long as x'' is feasible, and hence, $x' \succ^R x''$ is impossible.

Given the discussion above, we can define a binary relation $\rhd$ on X such that $x'' \rhd x'$ if $x' \in B_c^t$ for some $t \in \mathcal{T}$, $x'' \in A^t \backslash x'$, and $x' \not\succ^R x''$. Since we start from a data set collected from an RSM (TRSM) model, for every $x' \in B_c^t$, there exists at least one $x'' \in A^t \backslash x'$ with $x'' \rhd x'$ for which $x'' \succ' x'$ actually holds. Loosely speaking, $\rhd$ can be seen as a broad guess of the first step preference $\succ'$. In addition, the acyclicity of $\succ'$ requires that we can always find a selection $\rhd' \subset \rhd$ that is acyclic, and for every $t \in \mathcal{T}$ and $x' \in B_c^t$, there exists some $x'' \in A^t \backslash x'$ with $x'' \rhd' x'$. Furthermore, if the first step preference $\succ'$ is assumed to be transitive, a selection $\rhd'$ has to be chosen so that

$$\text{for every } x' \in B_c^t \text{ and } z^1, ..., z^k,\ x'' \rhd' z^1 \rhd' \cdots \rhd' z^k \rhd' x' \Longrightarrow x' \not\succ^R x''. \tag{19}$$

Now, $\rhd'$ is a "correct" guess of the first step preference, and if transitivity is imposed, the above implies that $x'' \succ' x'$. Hence, if $x' \succ^R x''$ were to hold, then it leads a contradiction that x' is deleted from a consideration set from which it is actually chosen. In fact, this observation is summarized in the conditions below, and plays a key role to characterize a data set that is rationalizable by an RSM (TRSM) model.

RSM-condition: Given a data set $\mathcal{O} = \{(a^t, A^t)\}_{t\in\mathcal{T}}$ containing Q revealed preference cycles, a traverse $c = (c_1, c_2, \ldots, c_Q) \in \times_{q=1}^{Q}$ obeys RSM-condition, if for the corresponding $\{B_c^t\}_{t\in\mathcal{T}}$, there exists an acyclic selection $\rhd'$ of $\rhd$, where for every $t \in \mathcal{T}$,

$$\text{for every } x' \in B_c^t, \text{ there exists } x'' \in A^t \text{ with } x'' \rhd' x'. \tag{20}$$

TRSM-condition: Given a data set $\mathcal{O} = \{(a^t, A^t)\}_{t\in\mathcal{T}}$ containing Q revealed preference

respect to the second step preference all other elements in $\Gamma(A^r) \subset \Gamma(A^s) \cup \Gamma(A^t)$.

cycles, a traverse $c = (c_1, c_2, \ldots, c_Q) \in \times_{q=1}^{Q}$ obeys TRSM-condition, if for the corresponding $\{B_c^t\}_{t \in \mathcal{T}}$, there exists an acyclic selection $\rhd'$ of $\rhd$ that obeys (19) and (20).

Theorem 4. *A data set $\mathcal{O} = \{(a^t, A^t)\}_{t \in \mathcal{T}}$ is rationalizable by an RSM model, if and only if there exists a traverse obeying RSM-condition.*

Theorem 5. *A data set $\mathcal{O} = \{(a^t, A^t)\}_{t \in \mathcal{T}}$ is rationalizable by a TRSM model, if and only if there exists a traverse obeying TRSM-condition.*

REMARK: In testing TRSM-condition, the search for an acyclic selection $\rhd'$ of $\rhd$ that obeys (19) and (20) can be done by way of a simple 0-1 integer programming (see Appendix II for the formulation). In the simulation and experiment in Section 6, we actually use it, which works very well. In principle, the RSM model $\rhd'$ can also be searched using a similar 0-1 integer programming. However, requiring acyclicity of $\rhd'$ in the programming can be computationally heavy, so applying 0-1 integer programming for RSM may not be practical.[14]

It is shown by Manzini and Mariotti (2007) that an RSM model can be characterized by a combination of two axioms on a data set, namely, Weak WARP and Expansion (see Appendix III). The former is implied when the consideration mapping obeys CFP. The latter requires that for every $A', A'' \subset X$, if $x = f(A') = f(A'')$, then $x = f(A' \cup A'')$, where f is a choice function. Given this, one may be tempted to consider that an RSM model is tested by the joint of CFP condition and (18), a straightforward partial-observation version of Expansion. The following example shows that this is not the case, i.e., we present a data set that obeys CFP condition and (18), but violates RSM condition. A similar example can be found for the joint of AFP+CFP condition and (18).

Example 3. *Let $X = \{x_1, x_2, x_3, x_4, x_5, x_6\}$ and consider a data set of six observations as below, where for every $t \in \mathcal{T}$, the chosen alternative is underlined.*

$$A^1 = \{\underline{x_1}, x_2, x_4\},\ A^2 = \{x_1, \underline{x_2}\},\ A^3 = \{\underline{x_3}, x_4, x_6\},\ A^4 = \{x_3, \underline{x_4}\},$$
$$A^5 = \{x_2, \underline{x_5}, x_6\},\ A^6 = \{x_5, \underline{x_6}\}.$$

It can be seen that Expansion, or (18), is trivially satisfied, because the chosen alternatives are all different. Note that there are four cycles with respect to $\succ^R$: $C_1 : x_1 \succ^R x_2 \succ^R x_1$, C_2 :

[14]For RSM, we applied a different strategy to find a suitable selection $\rhd'$, of which the detail is available from the authors upon request.

$x_3 \succ^R x_4 \succ^R x_3$, $C_3 : x_5 \succ^R x_6 \succ^R x_5$, and $C_4 : x_1 \succ^R x_4 \succ^R x_3 \succ^R x_6 \succ^R x_5 \succ^R x_2 \succ^R x_1$. We first show that RSM-condition cannot be satisfied by any traverse. Considering the first cycle, if we set $c_1 = (x_2, x_1)$, we will have $a^1 = x_1 \in B_c^2$. However, then, there does not exist any $x \in A^2$ such that $x_1 \nsucc^R x$, and we cannot define $\rhd$ for x_1. Therefore, we need to set $c_1 = (x_1, x_2)$ as the edge from the first cycle. By the same logic, we must set $c_2 = (x_3, x_4)$ and $c_3 = (x_5, x_6)$ regarding the second and third cycles respectively. Then we have $x_4 \rhd x_2$, $x_6 \rhd x_4$, and $x_2 \rhd x_6$, and it will be impossible to find an acyclic selection of $\rhd$. On the other hand, CFP-condition is satisfied by the traverse $c = ((x_1, x_2), (x_3, x_4), (x_5, x_6), (x_3, x_6))$. Note that the only set inclusions of feasible sets that we have are $A^t \subset A^{t-1}$ for $t = 2, 4, 6$. Meanwhile, $B_c^t = \varnothing$ for $t = 2, 4, 6$, so CFP-condition is trivially satisfied.

4.3 TRSM is weaker than WARP

When a choice function f is fully observed, as well known, it is rationalized by the rational choice model, if (and only if) $\succ^R$ is asymmetric, or the *weak axiom of revealed preference (WARP)* is satisfied. Hence, when a choice function is observable, any limited consideration model $\mathbf{M} \in$ {AFP, CFP, AFP+CFP, RSM, TRSM} is not refutable under WARP. However, for a general data set $\mathcal{O} = \{(a^t, A^t)\}_{t \in \mathcal{T}}$, WARP is no longer sufficient for $\mathcal{O}$ to be consistent with the rational choice model, and it is less obvious whether WARP is still sufficient for a data set to be consistent with all five models raised above. In the rest of this section, we show that, even for general data sets, WARP ensures the rationalizability by a TRSM model, and hence, any of the five models is not refutable.

Fix a data set $\mathcal{O} = \{(a^t, A^t)\}_{t \in \mathcal{T}}$. In order to rationalize it by a TRSM model, we need to find an asymmetric and transitive first step preference $\succ'$. In fact, if $\mathcal{O}$ obeys WARP, then we can find a traverse c, so that the transitive closure of S_c works as a first step preference. To get the idea, suppose that a data set generates $\succ^R$-relations as depicted in Figure 2. There, each arrow represents a revealed preference relation, i.e., $x^i \to x^j$ means that $x^i \succ^R x^j$. The list of cycles are: $C_1 : x^1 \succ^R x^2 \succ^R x^6 \succ^R x^1$, $C_2 : x^2 \succ^R x^3 \succ^R x^4 \succ^R x^2$, $C_3 : x^2 \succ^R x^6 \succ^R x^4 \succ^R x^2$, $C_4 : x^6 \succ^R x^4 \succ^R x^5 \succ^R x^6$, and $C_5 : x^1 \succ^R x^2 \succ^R x^3 \succ^R x^4 \succ^R x^5 \succ^R x^6 \succ^R x^1$. By letting $c = ((x^1, x^2), (x^4, x^2), (x^4, x^2), (x^4, x^5), (x^4, x^5)\}$, we have the corresponding binary relation $S_c = \{(x^1, x^2), (x^4, x^2), (x^4, x^5)\}$ (red arrows in the figure). Profile c is qualified as a traverse and the transitive closure of S_c (S_c itself in this case) works as an asymmetric and transitive first step preference. Indeed, by letting $x^1 \rhd x^2$, $x^4 \rhd x^2$, and $x^4 \rhd x^5$, then it is

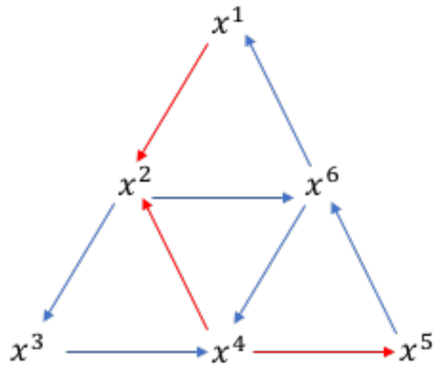

Figure 2: Adopting S_c as a first step preference.

acyclic and its transitive closure is $\rhd$ itself, and $x \rhd y$ implies that $y \not\succ^R x$. Hence, this $\rhd$ satisfies all requirements of the rationalizing first step preference for a TRSM model, and we can adopt it as $\succ'$.

The key trick in the above example is that traverse c is selected so that every cycle has at least two "unselected" edges. For instance, in the cycle $x^1 \succ^R x^2 \succ^R x^6 \succ^R x^1$, (x_2, x_6) and (x_6, x_1) are outside of S_c. As long as a traverse c is selected as such, we can always use the transitive closure of corresponding S_c as an asymmetric and transitive first step preference, which is shown in the proof of Theorem 6 below. Then, the issue is the existence of such a traverse, which is ensured by the following lemma.

Lemma 1. *Suppose that $\mathcal{O}$ obeys WARP. Then, we can find a traverse c so that every cycle has at least two unselected edges, or each cycle has at least two pairs of alternatives (x, y) such that $x \succ^R y$ and $(x, y) \notin S_c$.*

Theorem 6. *If a data set $\mathcal{O} = \{(a^t, A^t)\}_{t\in\mathcal{T}}$ obeys WARP, then it is rationalizable by a TRSM model.*

Since TRSM model is a special case of all other limited consideration models, namely AFP, CFP, AFP+CFP, and RSM models, this theorem yields the following corollary.

Corollary 1. *If a data set $\mathcal{O} = \{(a^t, A^t)\}_{t\in\mathcal{T}}$ obeys WARP, then AFP, CFP, AFP+CFP, RSM, and TRSM models are all non-refutable.*

5 Searching for a traverse

5.1 Essential cycles

As briefly mentioned in Section 3, given a model $\mathbf{M} \in \{\text{AFP, CFP, AFP+CFP, RSM, TRSM}\}$, it suffices to deal with the set of *essential* revealed preference cycles in searching for a traverse obeying $\mathbf{M}$-condition. Recall that a revealed preference cycle C is essential, if it does not contain any cycles except for itself. In what follows, suppose that a data set contains Q revealed preferences in total, and that amongst them, $\bar{Q}(\leqslant Q)$ cycles are essential. With no loss of generality, we may assume that cycles $\{C_1, C_2, ..., C_{\bar{Q}}\}$ are essential.

Lemma 2. *For every profile $\bar{c} = (\bar{c}_1, \bar{c}_2, ..., \bar{c}_{\bar{Q}}) \in \times_{q=1}^{\bar{Q}} C_q$, there exists a profile $c = (c_1, c_2, ..., c_Q) \in \times_{q=1}^{Q} C_q$ such that $S_c = \{\bar{c}_q\}_{q=1}^{\bar{Q}}$, i.e., for every $q \geqslant \bar{Q}+1$, there exists $\bar{q} \leqslant \bar{Q}$ with $c_q = \bar{c}_{\bar{q}}$.*

Proof. Recall that for each $q = \bar{Q}+1, \bar{Q}+2, ..., Q$, the cycle C_q is not essential, and hence, it contains an essential cycle $C_{\bar{q}}$ ($\bar{q} \leqslant \bar{Q}$). Then, it holds that $\{\bar{c}_{\bar{q}}\}_{\bar{q}=1}^{\bar{Q}} \cap C_q \neq \varnothing$ for each $q = \bar{Q}+1, \bar{Q}+2, ..., Q$. Choosing c_q from this intersection for $q \geqslant \bar{Q}+1$, we can extend $\bar{c}$ such that $c = (\bar{c}_1, \bar{c}_2, ..., \bar{c}_{\bar{Q}}; c_{\bar{Q}+1}, ..., c_Q)$, which obviously satisfies the desired property. □

We say that a traverse $c = (c_1, c_2, ..., c_Q)$ is *essential*, if there exists a profile of edges of essential cycles $(\bar{c}_1, \bar{c}_2, ..., \bar{c}_{\bar{Q}}) \in \times_{q=1}^{\bar{Q}} C_q$ that satisfies $S_c = \{\bar{c}_q\}_{q=1}^{\bar{Q}}$. The following proposition ensures that the set of essential traverses has sufficient information in testing $\mathbf{M}$-condition.

Proposition 2. *If there exists a traverse $c = (c_1, c_2, ..., c_Q)$ obeying $\mathbf{M}$-condition, then there also exists an essential traverse $c' = (c'_1, c'_2, ..., c'_Q)$ obeying $\mathbf{M}$-condition.*

Proof. Let $c = (c_1, c_2, ..., c_Q)$ be a traverse obeying $\mathbf{M}$-condition, and let $\bar{c} = (c_1, c_2, ..., c_{\bar{Q}}) \in \times_{q=1}^{\bar{Q}} C_q$. By Lemma 2, we can find an essential traverse $c' = (c'_1, c'_2, ..., c'_Q) \in \times_{q=1}^{Q} C_q$ such that $S_c = \{\bar{c}_q\}_{q=1}^{\bar{Q}}$, which actually obeys $\mathbf{M}$-condition. It is obvious that $S_{c'} \subset S_c$, and hence, $B_{c'}^t \subset B_c^t$ for every $t \in \mathcal{T}$. By the structure of $\mathbf{M}$-condition, we can see the following: whenever we have "larger" B^t-sets, (i) the LHS of AFP-condition is more permissive; (ii) the RHS of CFP-condition is more permissive; (iii) the LHS of AFP+CFP-condition is more permissive; and (iv) $\rhd$ is stronger and thus more difficult to find an acyclic (asymmetric and transitive) selection of it in RSM (TRSM)-condition. All of them imply that c' obeys $\mathbf{M}$-condition whenever c obeys $\mathbf{M}$-condition. □

Given this, in testing the rationalizability by **M**-model, it suffices to check the existence of an essential traverse obeying **M**-condition. Since each essential traverse is determined by a sequence of edges of essential cycles, we only need to deal with the set of essential cycles. In particular, for every essential traverse c, the set S_c is equal to the set of edges selected from essential cycles, and hence each B_c^t can be also calculated from a traverse *across essential cycles*. Provided that all **M**-conditions are written in terms of the properties of sets $\{B_c^t\}_{t\in\mathcal{T}}$, this ensures that we can focus on the set of essential cycles with no loss of generality.

5.2 Backtracking

The revealed preference tests in Sections 3 and 4, as well as De Clippel and Rozen (2014)'s AFP test, all involve combinatorial calculations, and applying them to actual data may be computationally challenging. However, the tests become manageable with the help of a simple but powerful method called *backtracking*.[15] In this section, we illustrate how this method is adopted to our revealed preference tests, after a brief introduction of this method.

To get the basic idea of backtracking, consider a problem where we have to choose c_q from some set C_q for every $q = 1, 2, ..., Q$, so that the resulting sequence $(c_1, c_2, \ldots, c_Q)$ obeys some constraint $\mathbf{P}_Q$. While there are $\prod_{q=1}^{Q} |C_q|$ logically possible trials that we must check, the backtracking procedure may lead us to a solution with much fewer trials, especially when $\mathbf{P}_Q$ has the *cut-off* property defined below. For every $\bar{Q} < Q$, let us refer to $(c_1, c_2, \ldots, c_{\bar{Q}})$ as a *partial* sequence in the sense that c_q is not yet determined for $q \in \{\bar{Q}+1, \ldots, Q\}$. Then, we say that $\mathbf{P}_Q$ has the cut-off property if: (I) for every $\bar{Q} < Q$, there exists a constraint $\mathbf{P}_{\bar{Q}}$, which is a length-$\bar{Q}$-modified version of $\mathbf{P}_Q$; and (II) partial sequence $(c'_1, c'_2, \ldots, c'_{\bar{Q}})$ violating $\mathbf{P}_{\bar{Q}}$ implies violation of $\mathbf{P}_{\bar{Q}+1}$ for any partial sequence $(c'_1, c'_2, \ldots, c'_{\bar{Q}}, c_{\bar{Q}+1})$. Given the cut-off property, if some partial sequence $(c'_1, c'_2, \ldots, c'_{\bar{Q}})$ violates $\mathbf{P}_{\bar{Q}}$, then there is no need to waste time on searching for subsequent components $c_{\bar{Q}+1}, \ldots, c_Q$, since there is no chance of any sequence $(c'_1, c'_2, \ldots, c'_{\bar{Q}}, c_{\bar{Q}+1}, \ldots, c_Q)$ satisfying $\mathbf{P}_Q$. In fact, this feature is at the heart of backtracking, and allows us to adopt a component-by-component search for a desired sequence.

Given below is a basic algorithm of the backtracking method. We consider a case where C_q is finite for every q, so with no loss of generality, we assume that sets C_q are a sets of integers.

[15]Some foundational references of the backtracking method are Walker (1960), Davis, Logemann, and Loveland (1962), and Golomb and Baumert (1965).

Basic backtracking algorithm. Given sets $(C_q)_{q=1}^{Q}$ and constraints $(\mathbf{P}_q)_{q=1}^{Q}$, this algorithm yields a sequence $(c_1, c_2, \ldots, c_Q)$ that satisfies $\mathbf{P}_Q$, or $\varnothing$ (meaning that $\mathbf{P}_Q$ cannot be satisfied).

1. [Initialize.] Set $\bar{Q} \leftarrow 0$.
2. [Enter level $\bar{Q}+1$.] (Now $(c_1, \ldots, c_{\bar{Q}})$ obeys $\mathbf{P}_{\bar{Q}}$.) Set $\bar{Q} \leftarrow \bar{Q}+1$. Then set $c_{\bar{Q}} \leftarrow \min C_{\bar{Q}}$.
3. [Test $(c_1, \ldots, c_{\bar{Q}})$.] If $(c_1, \ldots, c_{\bar{Q}})$ obeys $\mathbf{P}_{\bar{Q}}$, go to 6.
4. [Try again.] If $c_{\bar{Q}} \neq \max C_{\bar{Q}}$, set $c_{\bar{Q}}$ to the next larger element of $C_{\bar{Q}}$, and go to 3.
5. [Backtrack.] Set $c_{\bar{Q}} \leftarrow \min C_{\bar{Q}}$ and $\bar{Q} \leftarrow \bar{Q} - 1$. If $\bar{Q} = 0$, return $\varnothing$ and stop. Otherwise, go to 4.
6. [Terminate.] If $\bar{Q} = Q$, return $(c_1, \ldots, c_{\bar{Q}})$ and stop. Otherwise, go to 2.

The big picture of this algorithm is as follows. The process initially starts from considering a singleton sequence (c_1) and sees whether $\mathbf{P}_1$ is satisfied. If there is no such element in C_1, then we can immediately conclude that there is no chance of finding a sequence $(c_1, c_2, \ldots, c_Q)$ obeying $\mathbf{P}_Q$. If we find a successful partial sequence $(c_1, c_2, \ldots, c_{\bar{Q}-1})$ and reach the $\bar{Q}$-th level, we set $c_{\bar{Q}}$ to be the minimum element in $C_{\bar{Q}}$, and test whether $(c_1, c_2, \ldots, c_{\bar{Q}})$ obeys $\mathbf{P}_{\bar{Q}}$. If $\mathbf{P}_{\bar{Q}}$ is satisfied, then we proceed to the $(\bar{Q}+1)$-th level. If not, we redefine $c_{\bar{Q}}$ to be the next larger element of $C_{\bar{Q}}$ and check $\mathbf{P}_{\bar{Q}}$. If we cannot find any $c_{\bar{Q}} \in C_{\bar{Q}}$ such that $(c_1, c_2, \ldots, c_{\bar{Q}})$ obeys $\mathbf{P}_{\bar{Q}}$, then we go back to the $(\bar{Q}-1)$-th level and update $c_{\bar{Q}-1}$. This search algorithm terminates when we succeed in finding some $(c_1, c_2, \ldots, c_Q)$ obeying $\mathbf{P}_Q$, or it is determined that any (partial) sequence with $c_1 = \max C_1$ cannot be successful.

We now show that the backtracking method is applicable to our revealed preference tests as follows. Suppose that a data set $\mathcal{O} = \{(a^t, A^t)\}_{t \in \mathcal{T}}$ has Q essential revealed preference cycles. For each $q = 1, 2, ..., Q$, let C_q be the set of pairs $(x, y) \in \succ^R$ in the q-th revealed preference cycle. With a slight abuse of terminology, let us say that a sequence $(c_1, c_2, \ldots, c_Q) \in \times_{q=1}^{Q} C_q$ is *acyclic*, if its corresponding binary relation denoted by $S_Q = \{c_q\}_{q=1}^{Q}$ is acyclic. Then, note that such sequence $(c_1, c_2, ..., c_Q)$ is a traverse if it is acyclic. Hence, if we set $\mathbf{P}_Q$ as the joint of acyclicity and $\mathbf{M}$-condition, the revealed preference test for $\mathbf{M}$-model is equivalent to the existence problem of a sequence $(c_1, c_2, ..., c_Q)$ obeying constraint $\mathbf{P}_Q$. We claim that the above defined $\mathbf{P}_Q$ obeys the cut-off property for every $\mathbf{M} \in$ {AFP, CFP, AFP+CFP, RSM, TRSM}.

Condition (I): We define $\mathbf{P}_{\bar{Q}}$ for every $\bar{Q} \leqslant Q$ as follows. Given a partial sequence $(c_1, c_2, \ldots, c_{\bar{Q}})$, we can define the corresponding binary relation $S_{\bar{Q}} = \{c_q\}_{q=1}^{\bar{Q}}$. Since acyclicity of $(c_1, c_2, \ldots, c_{\bar{Q}})$ is defined via acyclicity of binary relation $S_{\bar{Q}}$, acyclicity is a well-defined constraint. Now we define a partial sequence version of **M**-condition, to which we refer as $\mathbf{M}_{\bar{Q}}$-condition as follows. Similar to (9), we can define for every $t \in \mathcal{T}$,

$$B^t_{\bar{Q}} = \{y \in A^t : y S^{-1}_{\bar{Q}} a^t\}. \tag{21}$$

We say that a partial sequence $(c_1, c_2, ..., c_{\bar{Q}})$ obeys $\mathbf{M}_{\bar{Q}}$-condition, if the corresponding $\{B^t_{\bar{Q}}\}_{t \in \mathcal{T}}$ satisfies the restriction in **M**-condition; specifically, $(c_1, c_2, ..., c_{\bar{Q}})$ obeys AFP$_{\bar{Q}}$-condition, if it holds that for every $s, t \in \mathcal{T}$,

$$\left[\left(A^s \backslash B^s_{\bar{Q}}\right) \cup \left(A^t \backslash B^t_{\bar{Q}}\right)\right] \subset (A^s \cap A^t) \Longrightarrow a^s = a^t. \tag{22}$$

Similar terminology is used for other models as well. With this $\mathbf{M}_{\bar{Q}}$-condition, we let $\mathbf{P}_{\bar{Q}}$ be the joint of acyclicity and $\mathbf{M}_{\bar{Q}}$-condition, which is clearly a well-defined constraint.

Condition (II): We show that if a partial sequence $(c_1, c_2, ..., c_{\bar{Q}})$ does not satisfy $\mathbf{P}_{\bar{Q}}$ for some $\bar{Q} < Q$, then $(c_1, c_2, ..., c_{\bar{Q}}, c_{\bar{Q}+1})$ cannot satisfy $\mathbf{P}_{\bar{Q}+1}$ for any $c_{\bar{Q}+1} \in C_{\bar{Q}+1}$. It is obvious, if $(c_1, c_2, ..., c_{\bar{Q}})$ is cyclic, then $(c_1, c_2, ..., c_{\bar{Q}}, c_{\bar{Q}+1})$ cannot be acyclic. Therefore, the substantial part is $\mathbf{M}_{\bar{Q}}$-condition. However, this follows straightforwardly by taking a look at our revealed preference conditions and the construction of B^t-sets. Note first that whenever $S_{\bar{Q}+1} \supset S_{\bar{Q}}$, it follows from (21) that $B^t_{\bar{Q}+1} \supset B^t_{\bar{Q}}$ for every $t \in \mathcal{T}$. Then following the same discussion as in the proof of Proposition 2, we conclude that $(c_1, c_2, ..., c_{\bar{Q}}, c_{\bar{Q}+1})$ fails $\mathbf{P}_{\bar{Q}+1}$ whenever $(c_1, c_2, ..., c_{\bar{Q}})$ fails $\mathbf{P}_{\bar{Q}}$.

Example 3 (continued)**.** *Consider the data set in Example 3. Let us walk through the backtracking algorithm, and see how we determine that the data set is not rationalizable by and RSM model. Recall that the data set has four cycles (we order the cycles and the edges in them as below):*

1. $C_1 = \{(x_1, x_2), (x_2, x_1)\}$,
2. $C_2 = \{(x_3, x_4), (x_4, x_3)\}$,
3. $C_3 = \{(x_5, x_6), (x_6, x_5)\}$,
4. $C_4 = \{(x_1, x_4), (x_4, x_3), (x_3, x_6), (x_6, x_5), (x_5, x_2), (x_2, x_1), \}$.

$\bar{Q}$	(partial) seq.	$\mathbf{P}_{\bar{Q}}$
1	$((x_1, x_2))$	PASS
2	$((x_1, x_2), (x_3, x_4))$	PASS
3	$((x_1, x_2), (x_3, x_4), (x_5, x_6))$	FAIL
3	$((x_1, x_2), (x_3, x_4), (x_6, x_5))$	FAIL
2	$((x_1, x_2), (x_4, x_3))$	FAIL
1	$((x_2, x_1))$	FAIL
0	$\varnothing$	**STOP**

Table 2: Backtracking procedure applied to Example 3 for testing RSM.

For every $\bar{Q} \in \{1, 2, 3, 4\}$, let us denote by $\mathbf{P}_{\bar{Q}}$ the joint of acyclicity and RSM$_{\bar{Q}}$-condition. Following our backtracking procedure, we first set $\bar{Q} = 1$ and set $c_1 = (x_1, x_2)$, which is the first edge of the first cycle. Since single element sequence $((x_1, x_2))$ obeys $\mathbf{P}_1$, we proceed to the second cycle by setting $\bar{Q} = 2$. Here we set $c_2 = (x_3, x_4)$ and check whether $((x_1, x_2), (x_3, x_4))$ obeys $\mathbf{P}_2$, which is affirmative. Then we go to the third cycle by setting $\bar{Q} = 3$ and set $c_3 = (x_5, x_6)$. In fact, this sequence $((x_1, x_2), (x_3, x_4), (x_5, x_6))$ fails to satisfy $\mathbf{P}_3$, specifically RSM$_3$-condition. In this case, we keep $\bar{Q} = 3$, and update c_3 to the next element in C_3, and set $c_3 = (x_6, x_5)$. Then, we test whether this updated sequence $((x_1, x_2), (x_3, x_4), (x_6, x_5))$ obeys $\mathbf{P}_3$, which is negative. At this point, we can determine that it is impossible to find a sequence (c_1, c_2, c_3, c_4) obeying RSM-condition and acyclicity as long as $(x_1, x_2), (x_3, x_4)$ are chosen from C_1, C_2 respectively. Thus we backtrack $\bar{Q}$ to 2, and update c_2 to (x_4, x_3). Looking at $((x_1, x_2), (x_4, x_3))$, it fails $\mathbf{P}_2$. Since there is no chance of success unless (x_1, x_2) is discarded from the sequence, we rewind $\bar{Q}$ to 1, and update c_1 to (x_2, x_1). Then we check whether $((x_2, x_1))$ obeys $\mathbf{P}_1$, which is negative. Then $\bar{Q}$ is set to 0 and the algorithm terminates, which means that the data set is not rationalizable by RSM model.

REMARK 1: One advantage of the backtracking approach is that we may be able to determine, at an early stage of the process of search, that a data set fails the test. Due to this feature, calculation time does depend on how we order the cycles. We suggest that the cycles are sorted so that shorter cycles come first: whenever $q' < q''$, q'-th cycle is weakly shorter than q''-th cycle. The cycles in Example 3 are sorted in this way. Whenever this takes too much calculation time, it seems natural to list "problematic" cycles first. Problematic cycles are those such that a (partial) sequence fails when adding an element at that cycle. This may allow us to determine that a data set fails the test at an early stage of the backtracking process (and we actually adopt this type of strategy).

Remark 2: Backtracking can be applied to De clippel and Rozen (2014)'s AFP test as well. Recall that their test requires the existence of an acyclic binary relation $\succ^*$ such that, for every $s, t \in \mathcal{T}$, with $a^s, a^t \in A^s \cap A^t$ and $a^s \neq a^t$,

$$\exists x \in A^s \backslash A^t : a^s \succ^* x \text{ or } \exists x \in A^t \backslash A^s : a^t \succ^* x. \tag{23}$$

Suppose there are $Q > 0$ pairs of observations (s, t) such that $a^s, a^t \in A^s \cap A^t$ and $a^s \neq a^t$. It can be seen that backtracking is applicable to De Clippel and Rozen's test, by letting $\mathbf{P}_Q$ be acyclicity, and for q-th pair (s, t), defining

$$C_q = \left\{(x'', x') : [x'' = a^s \text{ and } x' \in A^s \backslash A^t] \text{ or } [x'' = a^t \text{ and } x' \in A^t \backslash A^s]\right\}.$$

6 Simulation and Experiment

The purpose of this section is twofold: one is to compare relative strength of observable restrictions across models based on randomly generated data sets, and the other is to compare the measure of "plausibility" across models based on experimental data sets.

The former can be regarded as a version of Bronars' test in the context of limited consideration models, and one can measure the strength of observable restriction of each model by using its pass rate.[16] If we collect a sufficiently large number of random choices according to a uniform distribution, then the pass rate approximates the proportion of choices that are model-consistent to all logically possible choices. If this value is very close to 1, then the model in question is very hard to refute, or its observable restriction is weak.

As shown by Selten (1991) and Beatty and Crawford (2011), this measure of observable restriction plays a key role in considering the measure of plausibility of a model based on empirical or experimental data sets, which is nothing but our second issue in this section. Given empirical or experimental data sets, *Selten's index* evaluates a model by the difference of the pass rate calculated from actual data sets and the proportion of model-consistent choices to all logically possible choices. Practically, as in Beatty and Crawford (2011), Selten's index is calculated as the difference between pass rate based on actual data and that of randomly generated data sets. We could say that a model with a higher Selten's index is "better" than

[16]Bronars (1987) deals with the revealed preference test of the classical consumer theory. There, the fail rate of GARP on randomly generated data sets on randomly generated budgets is calculated.

that with a lower Selten's index, or intuitively, a "nice" model in terms of Selten's index is a model with higher pass rate and stronger observable restrictions.[17]

6.1 Simulation

We generated 10,000 random data sets with $|X| = 10, |\mathcal{T}| = 20, \min |A^t| = 2$, and $\max |A^t| = 8$. Firstly, we randomly generated 100 variations of feasible sets $\mathbb{A}_n := \{A_n^t\}_{t\in\mathcal{T}}$ for $n = 1, \ldots, 100$: fixing n, in generating each A_n^t, we set $|A_n^t| \in \{2, \ldots, 8\}$ following a uniform distribution over the set of natural numbers $\{2, \ldots, 8\}$, and then choose $|A_n^t|$ elements from X following a uniform distribution over X. We also require that $A_n^s \neq A_n^t$ for $s \neq t$. For each profile of feasible sets $\mathbb{A}_n = \{A_n^t\}_{t\in\mathcal{T}}$, a random choice data set $\{a_{i,n}^t\}_{t\in\mathcal{T}}$ is generated for $i = 1, \ldots, 100$: fixing n and i, $a_{i,n}^t$ is chosen following a uniform distribution over A_n^t for every $t \in \mathcal{T}$. Consequently we have a random choice data set $\mathcal{O}_{i,n} = \{(a_{i,n}^t, A_n^t)\}_{t\in\mathcal{T}}$ for $i = 1, \ldots, 100$, for which we apply our revealed preference tests. Note that we randomize feasible sets, as well as choices over them, since in general, observable restriction of a specific model depends on the structure of the feasible sets $\mathbb{A}_n = \{A_n^t\}_{t\in\mathcal{T}}$. For example, if $A^s \cap A^t = \varnothing$ for every $s, t \in \mathcal{T}$, then SARP is trivially satisfied, which implies that all five limited consideration models are non-refutable.

For each $\mathcal{O}_{i,n} = \{(a_{i,n}^t, A_n^t)\}_{t\in\mathcal{T}}$, we tested AFP, CFP, AFP+CFP, RSM, and TRSM, as well as SARP and WARP. We derived the pass rates for these tests under each profile of feasible sets, as well as the average pass rates of them over 100 profiles of feasible sets. In addition, we apply straightforward adaptations of existing full observation based tests to our partially observed data sets to see if they could approximate necessary and sufficient conditions. For example, as explained in Section 2.2, *if* an entire choice function is observed, AFP models is equivalent to the acyclicity of the binary relation $\succ_{\text{AFP}}$, and the binary relation itself can be defined even under partially observed data sets. Then, it may be of interest to what extent the acyclicity of $\succ_{\text{AFP}}$ works as an approximation of AFP-condition, partly because the former is much easier to check. The same argument applies to other models (see Appendix III for

[17] Selten's index has an axiomatization as follows. Let $m(\alpha, \beta) \in [-1, 1]$ be a measure of plausibility of a model that depends on the empirical pass rate, say, $\alpha \in (0, 1)$, and the proportion of model-consistent choices to all logically possible choices, say, $\beta \in (0, 1)$. Then, any $m(\cdot, \cdot)$ obeying the following axioms is an affine transformation of Selten's index $\alpha - \beta$: [Monotonicity] $m(1, 0) > m(0, 1)$, [Equivalence] $m(1, 1) = m(0, 0)$, and [Aggregability] $m(\lambda\alpha_1 + (1-\lambda)\alpha_2, \lambda\beta_1 + (1-\lambda)\beta_2) = \lambda m(\alpha_1, \beta_1) + (1-\lambda)m(\alpha_2, \beta_2)$. The first and second axioms determine how a measure should deal with extreme realizations of α and β, and the third axiom essentially implies that $m(\cdot, \cdot)$ is a cardinal measure.

test	our tests	full obsv. tests
SARP	0	0
WARP	0	0
AFP	0.9927	0.9954
CFP	0.6298	0.6298
AFP+CFP	0.0396	0.6176
RSM	0.0259	0.5083
TRSM	0.0006	0.5050

Table 3: Average pass rates.

details of full observation based tests).

In Table 3, the average pass rates of 100 different profiles of feasible sets (10,000 agents) are summarized. The left column gives the pass rates of revealed preference tests presented in our paper, and the right column gives the pass rates of the corresponding straightforward adaption of full observation version tests. The pass rate results show that AFP model is extremely permissive, letting more than 99% of the random agents pass the test, and CFP model is also quite permissive. On the other hand, we can say that concerning AFP+CFP, RSM, and TRSM models, observable restrictions are reasonably strong.[18] While AFP itself is hard to reject, combining it with another model strengthens observable restrictions drastically: AFP+CFP is much more restrictive than CFP, and the same holds for TRSM (AFP+RSM by Proposition 1) and RSM. Note that the agents can be partitioned into eight types: agent obeys either (i) TRSM; (ii) RSM and AFP+CFP but not TRSM; (iii) RSM but not AFP+CFP; (iv) AFP+CFP but not RSM; (v) CFP and AFP but neither AFP+CFP nor RSM; (vi) only CFP; (vii) only AFP; (viii) none of the models. Out of 10,000 agents, the distribution of agents' type is as follows: (i) 24 agents, (ii) 4 agents, (iii) 186 agents, (iv) 369 agents, (v) 5575 agents, (vi) 140 agents, (vii) 3576 agents, and (viii) 126 agents.What is striking is that while more than 60% of all agents passed both AFP and CFP, the pass rate of AFP+CFP model is significantly lower (lower than 0.04). Also, we can loosely say that for AFP+CFP, RSM, and TRSM models, the gaps between our tests and the full observation tests are large. Thus, in general, it is not plausible to use the full observation versions when dealing with partially observed data sets.

Figure 3 visually summarizes the distributions of pass rates for each test, where the hori-

[18] In Table 3, the pass rates of SARP and WARP are zero. While in theory there exist choice patterns consistent with them, there were none within our samples.

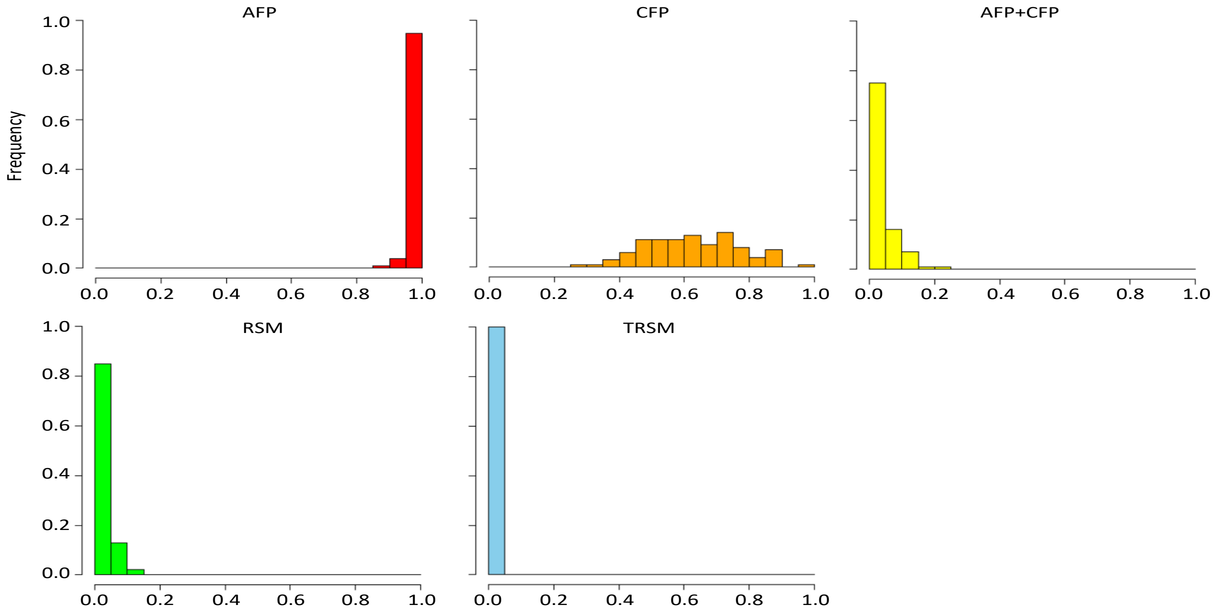

Figure 3: Histograms.

zontal axis is the pass rate given the profile of feasible sets, running from 0 to 1 with bin width 0.05. The vertical axis is the frequency of profiles of feasible sets of which the pass rates drop in each bin. It shows that the pass rate for CFP test has a large variance depending on the structure of feasible sets, while pass rates for other models are more accumulated to around either 0 and 1.

6.2 Experiment

We now proceed to the second issue of this section, or the comparison of Selten's indices based on experimental data sets. Amongst the profiles of feasible sets $\{\mathbb{A}_n\}_{n=1}^{100}$ generated for simulation, we chose one of them: one where the pass rates of the five limited consideration models are fairly "balanced." In choosing one profile of feasible sets, we first listed several of them where (i) pass rate of AFP is not 1 and (ii) pass rates of most models were distinct. Then for each of these profiles, we generated 1000 random choices, in order to assess the pass rates of each model in further detail. Finally, we picked one profile of feasible sets where pass rates of all five models were distinct, pass rate of AFP is not 1, and that of TRSM is not 0.

Following the experimental design of Manzini and Mariotti (2009), we consider the situation

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}
in 1 month	450	800	1150	450	450	800	850	1200	1550	500
in 3 months	800	800	800	450	1500	1150	0	0	0	0
in 5 months	1150	800	450	1500	450	450	1550	1200	850	1900

Table 4: The remuneration plans (in Japanese yen).

where each subject chooses remuneration plans of installments.[19] Each remuneration plan consists of 2400 Japanese yen overall, and this amount is split and installed in one month, three months, and five months after the experiment was conducted. Since the profile of feasible sets is one of those used in simulation, there are 10 alternatives and 20 feasible sets in total, and each feasible set consists of 2 – 8 alternatives (see Appendix IV for the contents of each feasible set). These numbers are not known to subjects. Most of the ten alternatives are in line with the eight alternatives used in Manzini and Mariotti (2009): there are "increasing," "constant," "decreasing," and "jump" series of payments and we added two "hump" payments in order to make the total number of alternatives ten. The alternatives are listed in Table 4: alternatives x_1 and x_7 are "increasing," x_2 and x_8 are "constant," x_3 and x_9 are "decreasing," x_4 and x_{10} are "jump," and x_5 and x_6 are "hump" series.[20]

The pass rate for each model is in the left column of Table 5. Statistical differences between the pass rates are significant at SARP/WARP & TRSM (at 1% significance level); AFP & CFP (5%); AFP & AFP+CFP (1%); and AFP+CFP & RSM (1%) in terms of a two-sample t-test assuming equal variance. The center column indicates random pass rates of models; to derive them, we generated 500,000 random choices over the feasible sets. Then, Selten's index of each model is derived as the difference between pass rates of experimental data and randomly generated data: for example Selten's index of AFP+CFP model is $0.8832 = 0.9115 - 0.0283$.

[19]Note that Manzini and Mariotti (2009) is the working paper version of Manzini and Mariotti (2012).

[20]The experiment was carried out at an experimental economics laboratory at the Faculty of Political Science and Economics, Waseda University, Japan. We ran 4 sessions and there was a total of 113 subjects. Subjects were recruited through an on-line bulletin that is accessible by all students. The proportion of male and female subjects were roughly the same. The experiment was computerized, and each participant was seated individually with a separator so that they cannot look at other participants' choices. Experimental sessions lasted an average of 42 minutes, of which the average duration of effective play was 11 minutes. The shortest session lasted 36 minutes and the longest 51 minutes. At the beginning of the experiment, subjects read instructions on paper, while the experimenter read the instructions aloud (see Appendix IV for an English translated version of the instructions). Preceding the remuneration-relevant stages, subjects were asked to take part in practice stages in order to be familiar with the usage of the computer in the experiment, and all subjects had to correctly answer questions that were asked to check whether the subjects understood the experimental design. It was explained that at the end of the experiment, one screen would be selected at random, and the chosen remuneration plan at that screen would be actually installed.

Looking at Table 5, for this experimental setting, AFP+CFP model distinctively well-performs in terms of Selten's index, while its pass rate is not significantly different from CFP model.[21]

	experiment pass rates	random pass rates	Selten's index
SARP	0.3363	0.0000	0.3363
WARP	0.3451	0.0000	0.3451
AFP	1.0000	0.9639	0.0361
CFP	0.9558	0.4714	0.4844
AFP+CFP	0.9115	0.0283	0.8832
RSM	0.5929	0.0157	0.5772
TRSM	0.5841	0.0012	0.5829

Table 5: Experimental pass rates, random pass rates, and Selten's indices.

Remark. Concerning the experimental pass rates, we also tested whether there are statistical differences between pass rates of subjects when we partition them with respect to decision time, sex, and reason of decision, using a two-sample t-test allowing different variances. There was no statistical difference between long-decision-time subjects and short-decision-time subjects; no statistical difference between male and female.[22] In a questionnaire following the experiment, we showed the subjects three experiment screens with their actual chosen alternatives indicated, and asked reasons of their choices. There were two clusters of subjects whose answers were consistent across these three decisions: one is a cluster of subjects who wanted to receive money "as soon as possible (a.s.a.p)" (29 subjects), and the other is the cluster of subjects who would like to receive money "as equally as possible through the three installments (smoothing)" (15 subjects). One may suspect these agents tend to be more rational, but there were no statistical differences between these 44 subjects and the others; the "a.s.a.p." subjects and the others; the "smoothing" subjects and the others; or the "a.s.a.p." subjects and the "smoothing" subjects.

We finally refer to the result of a comparative experiment, in which the cardinality of feasible sets vary from 2 to 5. The set of alternatives is the same as the baseline setting, and the number of feasible sets is also the same as before. The purpose of this experiment is to see how the size of feasible sets affects the comparison in terms of Selten's index. Similar to

[21]Concerning the random pass rates of SARP and WARP, similar to the case of Table 3, in theory there exist choices that are consistent with SARP, but there were none within the 500,000 randomly generated choices.

[22]A long-decision-time (short-decision-time) subject is a subject whose average decision time across 20 decisions is longer (shorter) than the median of all subjects.

	experiment pass rates	random pass rates	Selten's index
SARP	0.3875	0.0000	0.3875
WARP	0.4250	0.0000	0.4250
AFP	1.0000	0.9718	0.0282
CFP	0.9375	0.7987	0.1388
AFP+CFP	0.9125	0.3674	0.5451
RSM	0.6250	0.2038	0.4212
TRSM	0.6125	0.0623	0.5502

Table 6: Result of the comparative experiment.

the case of the baseline setting, we fix 20 feasible sets so that pass rates of five models are distinct, and pass rate of AFP model is not 1 and that of TRSM is not 0. This experiment was carried out at the same facility with the baseline experiment, and there was a total of 80 subjects in 3 sessions.[23]

The experimental pass rates, random pass rates, and Selten's indices are summarized in Table 6. Comparing the experimental pass rates of Tables 5 and 6, it seems that the pass rates of relatively rational models, namely RSM, TRSM, and SARP/WARP, are slightly higher in the comparative setting, where subjects choose from smaller feasible sets. However, there was no statistical significance in the difference of pass rates of each model across the two experiments. On the other hand, random pass rates are quite different across the two experiments, and hence Selten's indices differ as well. In this comparative experiment, we see that TRSM model explains subjects' behavior the best, and the explanatory power of AFP+CFP is not as high as in the baseline experiment, due to the fact that the random pass rate is higher in this comparative experiment. This shows that the explanatory power of models may vary drastically in different environments, even when the observed pass rates are similar.

Appendix I: Proofs

Proof of Proposition 1

We first show that if $(\succ, \Gamma)$ is a TRSM model, then it obeys AFP. To see this, suppose that $x \in A$ and $x \notin \Gamma(A)$. If $z \in \Gamma(A)$, there exists no $x' \in A$ such that $x' \succ' z$, and, in particular,

[23]The recruitment procedure of subjects is also same with the baseline experiment, and no subject in this comparative experiment participated in the baseline experiment, and vice versa.

there is no such x' in $A\backslash x$. Hence, it holds that $\Gamma(A) \subset \Gamma(A\backslash x)$. To see the converse set inclusion, suppose that $z \in \Gamma(A\backslash x)$, or there exists no $x' \in A\backslash x$ such that $x' \succ' z$. If $z \notin \Gamma(A)$ were to hold, it must be that $x \succ' z$. Since $x \notin \Gamma(A)$, there exists some $x' \in A\backslash x$ such that $x' \succ' x$. However, by transitivity, this implies that $x' \succ' z$, contradicting the assumption that $z \in \Gamma(A\backslash x)$. Hence, it holds that $z \in \Gamma(A)$, which, in turn, implies that $\Gamma(A\backslash x) \subset \Gamma(A)$.

Conversely, suppose that $(\succ, \Gamma)$ is an RSM model obeying AFP. Let $\succ'$ be the first step preference corresponding to $(\succ, \Gamma)$, and suppose that it is not transitive. Then, for some $x_1, x_2, x_3 \in X$, $x_1 \succ' x_2 \succ' x_3$ but $x_1 \not\succ' x_3$. However, this implies that $\Gamma(\{x_1, x_2, x_3\}) = \{x_1\} \neq \{x_1, x_3\} = \Gamma(\{x_1, x_3\})$, which contradicts AFP. Thus, $x_1 \succ' x_2 \succ' x_3 \Longrightarrow x_1 \succ' x_3$ must be satisfied.

Proof of Theorem 1

Since "only if" part has already been shown through derivation of AFP-condition, we deal with "if" part here. Take an arbitrary traverse c that obeys AFP-condition, and define consideration mapping $\Gamma : 2^X \to 2^X$ such that for every $A \subset X$,

$$
\begin{aligned}
\Gamma(A) &= A\backslash B_c^t, \text{ if } A^t\backslash B_c^t \subset A \subset A^t \text{ for some } t \in \mathcal{T} \\
&= A \quad \text{otherwise.}
\end{aligned} \tag{24}
$$

We first show that this Γ is well-defined and obeys AFP. To see the former, suppose $A^s\backslash B_c^s \subset A \subset A^s$ and $A^t\backslash B_c^t \subset A \subset A^t$ hold for some $s, t \in \mathcal{T}$. This implies that $\left[(A^s\backslash B_c^s) \cup (A^t\backslash B_c^t)\right] \subset (A^s \cap A^t)$, and AFP-condition requires that $a^s = a^t$. Gathering this together with $A \subset (A^s \cap A^t)$, which is implied by the initial assumption, we have

$$A \cap B_c^s = \{y \in A : yS_c^{-1}a^s\} = \{y \in A : yS_c^{-1}a^t\} = A \cap B_c^t.$$

Thus, we have $A\backslash B_c^s = A\backslash B_c^t$, and Γ is well-defined. To see that Γ obeys AFP, take any $A \subset X$ and $x \in A$ with $x \notin \Gamma(A)$. This implies that there exists $t \in \mathcal{T}$ such that $A^t\backslash B_c^t \subset A \subset A^t$ and $x \in B_c^t$. Note that by definition, $\Gamma(A) = A\backslash B_c^t$. Now consider the set $A\backslash x$. Since $x \in B_c^t$, it follows that $A^t\backslash B_c^t \subset A\backslash x \subset A^t$, and thus $\Gamma(A\backslash x) = (A\backslash x)\backslash B_c^t$. Recalling that $x \in B_c^t$, it follows that $\Gamma(A\backslash x) = (A\backslash x)\backslash B_c^t = A\backslash B_c^t = \Gamma(A)$, as desired.

Based on Γ defined as above, let us define a binary relation $\succ^*$ as follows: $x'' \succ^* x'$, if

$x'' = a^t$ for some $t \in \mathcal{T}$, $x' \in \Gamma(A^t)$, and $x'' \neq x'$. In fact, this binary relation is acyclic. To see this, suppose by way of contradiction that there is a cycle: $x^1 \succ^* x^2 \succ^* \cdots \succ^* x^L \succ^* x^1$. Note that $x'' \succ^* x'$ implies $x'' \succ^R x'$, which follows by the way these binary relations are defined. Therefore, the cycle above implies $x^1 \succ^R x^2 \succ^R \cdots \succ^R x^L \succ^R x^1$. Then, since c is a traverse, there exists an edge $(x^\ell, x^{\ell+1})$ contained in c. Obviously $x^\ell \succ^R x^{\ell+1}$ implies that for some $t \in \mathcal{T}$, $x^\ell = a^t$, and $x^{\ell+1} \in A^t$. On the other hand, gathering together $(x^\ell, x^{\ell+1}) \in S_c$, $x^\ell = a^t$ and $x^{\ell+1} \in A^t$, it holds that $x^{\ell+1} \in B_c^t$, which in turn implies $x^{\ell+1} \notin \Gamma(A^t)$. Hence it is impossible to have $x^\ell = a^t \succ^* x^{\ell+1}$, contradicting the hypothesis.

As the final step of the proof, we show that $a^t \in \Gamma(A^t)$ for every $t \in \mathcal{T}$, which in fact follows immediately from AFP. To see this, suppose not, i.e., there exists $s \in \mathcal{T}$ such that $A^s \backslash B_c^s \subset A^t \subset A^s$ and $a^t \in B_c^s$. However, this is impossible, since AFP requires $a^s = a^t$, which contradicts $a^t \in B_c^s$. Therefore, for every $t \in \mathcal{T}$, $a^t \in \Gamma(A^t)$, which in turn implies that a^t maximizes $\succ^*$ within $\Gamma(A^t)$. Since $\succ^*$ is acyclic, the transitive closure of it is asymmetric and transitive, and hence by Szpilrajn's theorem, it can be extended to a strict preference $\succ$ on X. Then, $(\succ, \Gamma)$ is an AFP model that rationalized the data set.

Proof of Theorem 2

Similar to the case of Theorem 1, we only show "if" part. Let c be a traverse that obeys CFP-condition, and define $\{B_c^t\}_{t \in \mathcal{T}}$ corresponding to c as (9), i.e., $B_c^t = \{y \in A^t : y S_c^{-1} a^t\}$ for every $t \in \mathcal{T}$. Define a consideration mapping Γ such that for every $A \subset X$,

$$\Gamma(A) = A \Big\backslash \bigcup_{t : A^t \subset A} B_c^t. \tag{25}$$

This Γ obeys CFP. To see this, consider $A', A'' \subset X$ such that $A' \subset A''$, and $x \in A'$ with $x \notin \Gamma(A')$. Then it suffices to show $x \notin \Gamma(A'')$. Note that $x \notin \Gamma(A')$ implies that there exist some $t \in \mathcal{T}$ such that $A^t \subset A'$ and $x \in B_c^t$. Since $A' \subset A''$, we clearly have $A^t \subset A''$, and it follows that $x \notin \Gamma(A'')$.

Next, define $\succ^*$ such that $x'' \succ^* x'$ if $x'' = a^t$, $x' \in \Gamma(A^t)$, and $x'' \neq x'$. The acyclicity of this relation can be proved similar to the case of Theorem 1. Assuming the existence of a cycle $x^1 \succ^* x^2 \succ^* \cdots \succ^* x^L \succ^* x^1$, it immediately implies the existence of a revealed preference cycle $x^1 \succ^R x^2 \succ^R \cdots \succ^R x^L \succ^R x^1$. Then, there exists an edge $(x^\ell, x^{\ell+1})$ that is contained in c, or $(x^\ell, x^{\ell+1}) \in S_c$. Since $x^\ell = a^t$ for some $t \in \mathcal{T}$, this implies that $x^{\ell+1} \in B_c^t$. By the definition of Γ, it holds that $x^{\ell+1} \notin \Gamma(A^t)$, and hence $x^\ell \not\succ^* x^{\ell+1}$, which is a contradiction.

Finally, we show that for every $t \in \mathcal{T}$, $a^t \in \Gamma(A^t)$. Suppose not. Then there exists some $s \in \mathcal{T}$ such that $A^s \subset A^t$ and $a^t \in B_c^s$. However, this is impossible, since CFP-condition requires that $a^t \notin B_c^s$. Summarizing, we have shown that $\succ^*$ is acyclic and a^t maximizes $\succ^*$ within the set $\Gamma(A^t)$ for every $t \in \mathcal{T}$. Since $\succ^*$ is acyclic, the transitive closure of $\succ^*$ is asymmetric and transitive, and hence by Szpilrajn's theorem, it can be extended to a strict preference $\succ$ on X. In addition, since for every $t \in \mathcal{T}$, a^t maximizes $\succ^*$ within $\Gamma(A^t)$, it holds that $a^t \succ x$ for every $x \in \Gamma(A^t)\backslash a^t$. Summarizing, we conclude that the data set is rationalizable by limited consideration model $(\succ, \Gamma)$, where Γ obeys CFP.

Proof of Theorem 3

Similar to the preceding theorems, we construct a pair of a consideration mapping and a strict preference that rationalizes $\mathcal{O}$ based on a traverse c (and the corresponding $\{B_c^t\}_{t\in\mathcal{T}}$) obeying AFP+CFP-condition. To define Γ, we need the following set of indices for every $A \subset X$:

$$\tau(A) = \max\left\{\tau \subset \mathcal{T} : \bigcup_{r\in\tau} A^r \Big\backslash \bigcup_{r\in\tau} B_c^r \subset A\right\}. \tag{26}$$

Then, by using $\tau(A)$, define Γ such that

$$\Gamma(A) = A \Big\backslash \bigcup_{r\in\tau(A)} B_c^r. \tag{27}$$

Obviously, in order for the above definition to be well-defined, $\tau(A)$ must be uniquely determined for every $A \subset X$, which is actually the case. To see this, suppose to the contrary: there exist $\tau_1(A) \neq \tau_2(A)$ that obey (26). Then, we have $\left(\bigcup_{r\in\tau_1(A)} A^r \backslash \bigcup_{r\in\tau_1(A)} B_c^r\right) \subset A$ and $\left(\bigcup_{r\in\tau_2(A)} A^r \backslash \bigcup_{r\in\tau_2(A)} B_c^r\right) \subset A$, which implies that

$$\left[\bigcup_{r\in\tau_1(A)\cup\tau_2(A)} A^r \Big\backslash \left(\bigcup_{r\in\tau_1(A)} B_c^r \cup \bigcup_{r\in\tau_2(A)} B_c^r\right)\right] \subset A.$$

Obviously, this can be rewritten as

$$\left(\bigcup_{r\in\tau_1(A)\cup\tau_2(A)} A^r \Big\backslash \bigcup_{r\in\tau_1(A)\cup\tau_2(A)} B_c^r\right) \subset A.$$

By defining $\tau(A) = \tau_1(A) \cup \tau_2(A)$, we have $\tau(A) \supsetneq \tau_i(A)$ for $i = 1, 2$, which contradicts the maximality of $\tau_1(A)$ and $\tau_2(A)$.

Given that Γ defined as (27) is well-defined, we move on to show that it obeys both AFP and CFP. Consider any $A', A'' \subset X$ with $A' \subset A''$, and $x \in A'$ such that $x \notin \Gamma(A')$. This means that $x \in \bigcup_{r \in \tau(A')} B_c^r$. Since $\tau(\cdot)$ is clearly monotonic, it follows that $\tau(A') \subset \tau(A'')$, and hence, $x \in \bigcup_{r \in \tau(A'')} B_c^r$. This assures that $x \notin \Gamma(A'')$, which shows CFP. To see AFP, take any $A \subset X$ and any $x \in A$ with $x \notin \Gamma(A)$. This means that $x \in \bigcup_{r \in \tau(A)} B_c^r$, which in turn implies that

$$\left(\bigcup_{r \in \tau(A)} A^r \Big\backslash \bigcup_{r \in \tau(A)} B_c^r \right) \subset A \backslash x. \tag{28}$$

The maximality and uniqueness of $\tau(\cdot)$, combined with (28), imply $\tau(A) \subset \tau(A \backslash x)$. On the other hand, the monotonicity of $\tau(\cdot)$ implies $\tau(A \backslash x) \subset \tau(A)$. Hence we have $\tau(A) = \tau(A \backslash x)$. Then, we have $\Gamma(A \backslash x) = (A \backslash x) \backslash \bigcup_{r \in \tau(A \backslash x)} B_c^r = A \backslash \bigcup_{r \in \tau(A)} B_c^r = \Gamma(A)$, which is the desired result.

Let $\succ^*$ be a binary relation such that $x'' \succ^* x'$, if $x'' = a^t$, $x' \in \Gamma(A^t)$, and $x'' \neq x'$. We show that $\succ^*$ is acyclic, and thus extendable to a strict preference. By way of contradiction, suppose that there exists a cycle $x^1 \succ^* x^2 \succ^* \cdots \succ^* x^L \succ^* x^1$, which clearly implies $x^1 \succ^R x^2 \succ^R \cdots \succ^R x^L \succ^R x^1$. Then, there exists an edge $(x^\ell, x^{\ell+1})$ contained in c, or more precisely, S_c. Since $x^\ell = a^t$ and $x^{\ell+1} \in A^t$ hold for some $t \in \mathcal{T}$, this means that $x^{\ell+1} \in B_c^t$ for such an observation t. It is easy to check from the definition of Γ that $t \in \tau(A^t)$, and hence, $x^{\ell+1} \notin \Gamma(A^t) \subset A^t \backslash B^t$. However, then, it holds that $x^\ell \not\succ^* x^{\ell+1}$, which is a contradiction.

Finally, let us show that $a^t \in \Gamma(A^t)$ for every $t \in \mathcal{T}$, which follows immediately from AFP+CFP-condition. Indeed, for every $t \in \mathcal{T}$, we have $\left(\bigcup_{r \in \tau(A^t)} A^r \backslash \bigcup_{r \in \tau(A^t)} B_c^r \right) \subset A^t$, and then, AFP+CFP-condition requires $a^t \notin \bigcup_{r \in \tau(A^t)} B_c^r$, which in turn ensures $a^t \in \Gamma(A)$ for every $t \in \mathcal{T}$. Since $\succ^*$ is acyclic, it is extendable to a strict preference $\succ$ on X using Szpilrajn's theorem. Then this $\succ$ and Γ defined as (27) combined together is an AFP+CFP model that rationalizes the data set.

Proof of Theorems 4 and 5

The proofs of Theorems 4 and 5 are almost identical, so we provide the proofs of them jointly. Since the necessity parts of them have been already discussed, we prove the sufficient parts of them based on a traverse obeying RSM (TRSM)-condition. Using an acyclic selection $\rhd'$ of $\rhd$, define Γ as

$$\Gamma(A) = \{x \in A : \nexists x' \in A \text{ such that } x' \rhd' x\}. \tag{29}$$

Note that the selection $\rhd'$ is acyclic, so we use it as a first step preference for the case of Theorem 4. If we can find $\rhd'$ so that it obeys (19) in addition to (20), then we use the transitive closure of it, say, $\rhd''$ as a first step preference and define Γ by using it instead of $\rhd'$. Note further that $\Gamma(A^t) \subset A^t \backslash B_c^t$ holds, by the definition of $\rhd'$ (or $\rhd''$) and the construction of Γ. The remaining substantial parts of the proof are to show that $a^t \in \Gamma(A^t)$ for every $t \in \mathcal{T}$, and the binary relation $\succ^*$ defined as $x'' \succ^* x'$ if $x'' = a^t, x' \in \Gamma(A^t)$, and $x'' \neq x'$ is acyclic.

To prove that $\succ^*$ is acyclic, suppose to the contrary, i.e., there is a cycle: $x^1 \succ^* x^2 \succ^* \cdots \succ^* x^L \succ^* x^1$. Since we have $\succ^* \subset \succ^R$, this cycle implies $x^1 \succ^R x^2 \succ^R \cdots \succ^R x^L \succ^R x^1$. Then, since c is a traverse, there exists an edge $(x^\ell, x^{\ell+1})$ contained in c, and we have $x^{\ell+1} \in B_c^t$ for every $t \in \mathcal{T}$ with $x^\ell = a^t$ and $x^{\ell+1} \in A^t$. By RSM (TRSM)-condition, there exists some $x \in A^t$ such that $x \rhd' (\rhd'') x^{\ell+1}$, which in turn implies $x^{\ell+1} \notin \Gamma(A^t)$. Then it is impossible to have $x^\ell = a^t \succ^* x^{\ell+1}$, and we conclude that $\succ^*$ is acyclic.

To see that $a^t \in \Gamma(A^t)$ for every $t \in \mathcal{T}$, by way of contradiction, suppose that for some $t \in \mathcal{T}$, $a^t \notin \Gamma(A^t)$. This means that there exists $x \in A^t \backslash a^t$ such that $x \rhd' a^t$, which in turn implies $x \rhd a^t$. However, this is not possible, since $x \rhd a^t$ requires $a^t \nsucc^R x$, while we have $a^t \succ^R x$. When a traverse obeys TRSM-condition and Γ is defined as the set of maximal elements with respect to $\rhd''$, $a^t \notin \Gamma(A^t)$ implies the existence of some $x \in A^t \backslash a^t$ such that $x \rhd'' a^t$. However, this is also impossible, since $x \rhd'' a^t$ implies the existence of a sequence $z^1, z^2, ..., z^k$ such that $x \rhd' z^1 \rhd' \cdots \rhd' z^k \rhd' a^t$, and by TRSM-condition, $a^t \nsucc^R x$, which contradicts the assumption that $x \in A^t$. The rest of the proof is to extend the transitive closure of $\succ^*$ to a strict preference by using Szpilrajn's theorem. Then it can easily be seen that the data set is rationalized by an RSM (TRSM) model $(\Gamma, \succ)$.

Proof of Lemma 1

Suppose that data set $\mathcal{O} = \{(a^t, A^t)\}_{t\in\mathcal{T}}$ contains Q revealed preference cycles in total. We construct a traverse $c = (c_1, c_2, \ldots, c_Q)$ via the following algorithm.[24] In the procedure, let us refer to a revealed preference cycle where an element for c has not been specified as an *untreated* cycle. In addition, if $x \succ^R y$ holds, then x and y are respectively referred to as the *source* and the *sink* of the edge (x, y).

1. Fix an arbitrary untreated cycle and an arbitrary alternative x from it.
2. For every untreated cycle containing x, choose an edge whose source is x as an edge for c.
3. Let Y_x be the set of alternatives that are sinks of the edges selected in 2. For every untreated cycle containing $y \in Y_x$, choose an edge whose sink is y as an edge for c.
4. Stop if there is no untreated cycle. Otherwise go to 1.

Since there are at most finite cycles, the algorithm stops in finitely many repetitions. Let $c = (c_1, c_2, \ldots, c_Q)$ be the profile of $\succ^R$-edges generated in the algorithm, and let $S_c = \{c_q\}_{q=1}^Q$ be the corresponding binary relation. It is obvious that every revealed preference cycle has at least one element of S_c. Take an arbitrary revealed preference cycle $x^{i_1} \succ^R x^{i_2} \succ^R \cdots \succ^R x^{i_1}$ and suppose that $(x^{i_k}, x^{i_{k+1}}) \in S_c$. We claim that, by the second and third steps of the algorithm respectively, the edges $(x^{i_{k-1}}, x^{i_k})$ and $(x^{i_{k+1}}, x^{i_{k+2}})$ cannot be selected as elements of S_c, which also ensures the acyclicity of S_c. We only show that $(x^{i_{k-1}}, x^{i_k}) \notin S_c$, since the proof for the other case is similar. Suppose that $(x^{i_k}, x^{i_{k+1}})$ is added to c in the r-th repetition of the algorithm. Then, the cycle $x^{i_1} \succ^R x^{i_2} \succ^R \cdots \succ^R x^{i_1}$ must have been untreated until then, which implies that $(x^{i_{k-1}}, x^{i_k})$ cannot have been added to c before r-th repetition. It is also impossible for it to be added to c at r-th repetition or later as follows. If a cycle contains $(x^{i_{k-1}}, x^{i_k})$, then it also contains an edge in the form of (x^{i_k}, y). Hence, if such a cycle is untreated at r-th repetition, the edge (x^{i_k}, y) would be selected by the second step of the algorithm, and there is no chance for $(x^{i_{k-1}}, x^{i_k})$ to be selected at r-th repetition. This also implies that, after r-th repetition, any untreated cycle does not contain $(x^{i_{k-1}}, x^{i_k})$, and hence it cannot be added to c there. Summarizing, we must have $(x^{i_{k-1}}, x^{i_k}) \notin S_c$.

[24] One can confirm that c in the example in Section 4.4 is specified via this algorithm with starting from $x^1 \succ x^2 \succ x^6 \succ x^1$.

Proof of Theorem 6

If $\mathcal{O}$ contains no revealed preference cycle, then it is obviously rationalizable by a TRSM model. Suppose that $\mathcal{O}$ obeys WARP, or every cycle associated with it consists of more than two alternatives. By Lemma 1, we can find a traverse c such that every cycle has at least two unselected edges. Fix such a traverse c and set $\rhd = S_c$, or $x \rhd y \iff xS_c y$. Then, since S_c is an acyclic selection of $\succ^R$, it holds that $x \rhd y \Longrightarrow x \succ^R y$ and that $\rhd$ is acyclic. Hence, by letting $\rhd'$ be the transitive closure of $\rhd$, it is asymmetric and transitive. In addition, it holds that $x \rhd' y \Longrightarrow y \not\succ^R x$. Indeed, since $\rhd = S_c \subset \succ^R$, if we have a sequence like $x \rhd z^1 \rhd \cdots \rhd z^k \rhd y \succ^R x$, this means that there exists a revealed preference cycle where only one edge is unselected, contradicting our hypothesis. This ensures that for every $t \in \mathcal{T}$, there is no $x \in A^t$ such that $x \rhd' a^t$. Given this $\rhd'$, we can construct a TRSM model by defining Γ and $\succ^*$ in the same way as the proof of Theorem 5.

Appendix II: Formulation of integer programming

Here we describe how we can formulate AFP test in De Clippel and Rozen (2014) and our RSM/TRSM test as 0-1 integer programming problems. Let us denote the integer problems as $\Theta \cdot \mathbf{x} \geqslant \mathbf{b}$, where matrix Θ and vector $\mathbf{b}$ are parameters determined from the data set and/or a traverse, and vector $\mathbf{x}$ is the vector of interest. Throughout this appendix, $\mathbf{x}$ is restricted to be a 0-1 vector.

Before presenting the integer programming formulation of De Clippel and Rozen's AFP test, we note again the statement of their result.

Theorem [De Clippel and Rozen] *A data set $\mathcal{O} = \{(a^t, A^t)\}_{t\in\mathcal{T}}$ is rationalizable by an AFP model if and only if there exists a binary relation $\succ^*$ on X such that*

(I) for every $s, t \in \mathcal{T}$ such that $a^s, a^t \in A^s \cap A^t$ and $a^s \neq a^t$,

$$\exists x' \in A^s \backslash A^t : a^s \succ^* x' \text{ or } \exists x'' \in A^t \backslash A^s : a^t \succ^* x'', \tag{30}$$

(II) binary relation $\succ^*$ is acyclic.

In the problem $\Theta \cdot \mathbf{x} \geqslant \mathbf{b}$, the matrix Θ and vector $\mathbf{b}$ are the factors for (I), and the acyclicity of $\succ^*$ is required through additional constraints on the solution vector $\mathbf{x}$. Specifically, vector $\mathbf{x} = (x_{11}, x_{21}, \dots, x_{n1}, \dots, x_{1n}, \dots, x_{nn})$ is interpreted as a vector that represents binary relation $\succ^*$: $x_{ij} = 1$ if $x_i \succ^* x_j$ and $x_{ij} = 0$ otherwise. As described below, matrix Θ and vector $\mathbf{b}$ are determined once data set $\mathcal{O}$ is observed. Let I be a set of index-pairs $(s, t) \in \mathcal{T} \times \mathcal{T}$ such that $a^s, a^t \in A^s \cap A^t$ and $a^s \neq a^t$, and let $|I| = m$. Then, Θ is a matrix with m rows ($\mathbf{b}$ is an m dimensional vector), where each row (entry) represents the requirements that assure (30). Fix any row, say k-th row, and suppose that indices $s, t \in \mathcal{T}$ are the indices associated with this row. The k-th row of Θ is an n^2 dimensional vector, which we denote as $\boldsymbol{\theta}_k = (\theta_{11}, \theta_{21}, \dots, \theta_{n1}, \dots, \theta_{1n}, \dots, \theta_{nn})$, and b_k (k-th entry of $\mathbf{b}$) is a scalar. We omit the index k from entries of $\boldsymbol{\theta}_k$ for the sake of notational simplicity. Given a data set, $\boldsymbol{\theta}_k$ is defined so that $\theta_{ij} = 1$ if (i) $x_i = a^s$ and $x_j \in A^s \backslash A^t$, or (ii) $x_i = a^t$ and $x_j \in A^t \backslash A^s$; and $\theta_{ij} = 0$ otherwise. That is, x_i corresponds to a^s (resp. a^t), and x_j corresponds to x' (resp. x'') in (I). Then, for (30) to hold, we must have $\boldsymbol{\theta}_k \cdot \mathbf{x} \geqslant 1$, so we can set $b_k = 1$.

The additional constraints that require acyclicity of $\succ^*$ are straightforward: for every cyclic sequence of indices $J = (i, j, k, \dots, \ell, i)$,

$$x_{ij} + x_{jk} + \cdots + x_{\ell i} \leqslant |J| - 2. \tag{31}$$

While these acyclicity constraints are easy to understand, since we must write a constraint for *every* cyclic sequence of indices, it may be computationally tough to list up: the number of constraints explodes as the number of alternatives gets larger.

Example 4. *Let $X = \{x_1, x_2, x_3, x_4\}$, and consider a data set of three observations as below:*

$$A^1 = \{\underline{x_1}, x_2, x_3, x_4\},\ A^2 = \{x_1, \underline{x_2}, x_3\},\ A^3 = \{x_2, \underline{x_3}, x_4\}.$$

Note that we have $a^1, a^2 \in A^1 \cap A^2$, $a^1 \neq a^2$ and $a^2, a^3 \in A^2 \cap A^3$, $a^2 \neq a^3$. Hence the matrix Θ has two rows, where the first row corresponds to observations $(1, 2)$, and the second row corresponds to observations $(2, 3)$. As for observations $(1, 2)$, $A^1 \backslash A^2 = \{x_4\}$ and $A^2 \backslash A^1 = \varnothing$, so we must have $a^1 = x_1 \succ^* x_4$, and thus the θ_{14} entry of $\boldsymbol{\theta}_1$ is 1. As for observations $(2, 3)$, $A^2 \backslash A^3 = \{x_1\}$ and $A^3 \backslash A^2 = \{x_4\}$, so we must have $a^2 = x_2 \succ^* x_1$ or $a^3 = x_3 \succ^* x_4$. Hence the entries θ_{21} and θ_{34} of $\boldsymbol{\theta}_2$ is 1. This is summarized in Table 7. In this example with 4 alternatives, we only need to list up 25 constraints regarding acyclicity of $\succ^*$.[25] *This number*

[25]There are 4 constraints that require asymmetry, 6 constraints regarding cycles involving two alternatives, 9 constraints regarding three-alternative cycles, and 6 regarding four-alternative cycles.

	θ_{11}	θ_{21}	θ_{31}	θ_{41}	θ_{12}	θ_{22}	θ_{32}	θ_{42}	θ_{13}	θ_{23}	θ_{33}	θ_{43}	θ_{14}	θ_{24}	θ_{34}	θ_{44}	$\mathbf{b}$
$\boldsymbol{\theta}_1$	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	1
$\boldsymbol{\theta}_2$	0	1	0	0	0	0	0	0	0	0	0	0	0	0	1	0	1

Table 7: Matrix Θ and vector b defined for De Clippel and Rozen's test in Example 4.

will explode as the number of alternatives gets larger.

In testing RSM/TRSM, we search for a traverse under which there exists an appropriate selection $\rhd'$ of binary relation $\rhd$. Recall that once a traverse c is determined, binary relation $\rhd$ is defined: $x'' \rhd x'$ if $x^t \in B_c^t$ for some $t \in \mathcal{T}$, $x'' \in A^t \backslash x'$, and $x' \nsucc^R x''$. We need to check if there exists an acyclic (or asymmetric and transitive) selection $\rhd'$ such that for every $x' \in B_c^t$, there exists $x'' \in A^t$ with $x'' \rhd' x'$. That is, $\rhd'$ has to be chosen so that every alternative in B_c^t is dominated by some other alternative in A^t.

By nature of the problem, similar to the case of De Clippel and Rozen, it can be rephrased as the solvability of a 0-1 integer problem $\Theta{\cdot}\mathbf{x} \geqslant \mathbf{b}$, and the restriction of acyclicity (asymmetry and transitivity) is required through some additional linear constraints. The solution vector $\mathbf{x} = (x_{11}, x_{21}, \ldots, x_{n1}, \ldots, x_{1n}, \ldots, x_{nn})$ is interpreted as a vector version of a selection $\rhd'$ from $\rhd$: $x_{ij} = 1$ if $x_i \rhd' x_j$, and $x_{ij} = 0$ otherwise, and matrix Θ is an $(nT \times n^2)$ matrix that tells us candidates of where to define $\rhd'$. More specifically, the matrix Θ consists of $(n \times n^2)$-matrices $\{\Theta^t\}_{t=1}^T$, and the vector $\mathbf{b}$ consists of n-dimensional vectors $\{\mathbf{b}^t\}_{t=1}^T$. For $i \in \{1, \ldots, n\}$, i-th row of Θ^t and i-th coordinate of $\mathbf{b}^t$ correspond to information regarding alternative x_i at t-th observation. Denote them by $\boldsymbol{\theta}_i^t = (\theta_{11}, \ldots, \theta_{n1}, \ldots, \theta_{1i}, \ldots, \theta_{ni}, \ldots, \theta_{1n}, \ldots, \theta_{nn})$ and b_i^t. Though every entry θ_{jk} of $\boldsymbol{\theta}_i^t$ depends on $t \in \mathcal{T}$ and $i \in \{1, 2, ..., n\}$, we omit them for the sake of notational simplicity. By using these notions, the problem $\Theta \cdot \mathbf{x} \geqslant \mathbf{b}$ is equivalent to $\boldsymbol{\theta}_i^t \cdot \mathbf{x} \geqslant b_i^t$, or $\sum_{j=1}^n \theta_{ji} x_{ji} \geqslant b_i^t$ for every $t \in \mathcal{T}$ and $i \in \{1, 2, ..., n\}$.

For every $t \in \mathcal{T}$ and $i \in \{1, 2, ..., n\}$, the entries of $\boldsymbol{\theta}_i^t$ and b_i^t are set to 0 except for the following cases.

(I) Suppose that $x_j \not\rhd x_k$. Since $\rhd'$ is defined as a selection from $\rhd$, we cannot have $x_j \rhd' x_k$, or equivalently $x_{jk} = 0$ must hold in such a case. To require this, for such a pair of indices (j, k), we let $\theta_{jk} = -n$.

(II) Suppose that $x_i \in B_c^t$, where B_c^t is specified by a given traverse c. Then, $x_{ji} = 1$ must hold for at least one j such that $x_j \rhd x_i$. To require this, we set $\theta_{ji} = 1$ for all such j

and $b_i^t = 1$.

Recall that for RSM model, this binary relation $\rhd'$ has to be acyclic, and for TRSM model it has to be asymmetric and transitive. These requirements will be made as constraints on the solution vector $\mathbf{x}$. RSM model requires that binary relation $\rhd'$ is acyclic, which requires $\mathbf{x}$ to satisfy (31). TRSM model requires that binary relation $\rhd'$ is asymmetric and transitive. These two constraints are assured as follows: for every $i, j, k \in \{1, \ldots, n\}$,

$$1 - x_{ij} - x_{ji} \geqslant 0, \tag{32}$$

$$x_{ij} + x_{jk} \leqslant 2x_{ik} + 1. \tag{33}$$

Constraint (32) assures asymmetry of $\rhd'$ and (33) assures transitivity of $\rhd'$.

It is not difficult to check that, by constructing Θ and $\mathbf{b}$ as above, a data set is rationalizable by an RSM model if and only if there exists a traverse c such that the problem $\Theta \cdot \mathbf{x} \geqslant \mathbf{b}$ has a solution $\mathbf{x}$ subject to constraint (31). A data set is rationalizable by a TRSM model if and only if there exists a traverse c such that the problem $\Theta \cdot \mathbf{x} \geqslant \mathbf{b}$ has a solution $\mathbf{x}$ subject to constraints (32) and (33).

Example 4 (continued)**.** *Note that there are three cycles with respect to* $\succ^R$*:* $x_1 \succ^R x_2 \succ^R x_1$*;* $x_2 \succ^R x_3 \succ^R x_2$*; and* $x_1 \succ^R x_3 \succ^R x_2 \succ^R x_1$*. Let* $c = ((x_1, x_2), (x_2, x_3), (x_1, x_3))$*, which implies* $B_c^1 = \{x_2, x_3\}$*,* $B_c^2 = \{x_3\}$*,* $B_c^3 = \varnothing$*, and binary relation* $\rhd$ *is such that:* $x_4 \rhd x_2$ *and* $x_1 \rhd x_3$*. Then the* Θ *matrix and* b *vector is defined as in Table 8.*
This problem $\Theta \cdot \mathbf{x} \geqslant \mathbf{b}$ *has a solution* $\mathbf{x} = (0, 0, 0, 0, 0, 0, 0, 1, 1, 0, 0, 0, 0, 0, 0, 0)$*, where* $x_{42}, x_{13} = 1$*, and* 0 *elsewhere. This means that by setting* $x_4 \rhd' x_2$ *and* $x_1 \rhd' x_3$*, which is obviously asymmetric and transitive, this data set is consistent with a TRSM model.*

Appendix III: Full observation tests

Here we introduce full observation version characterizations of the limited consideration models, and describe how we adapt them to the limited data context in our simulation. The full observation characterizations are based on observation of a choice function $f : 2^X \to X$, where $f(A) \in A$ for every $A \subset X$.

AFP, CFP, and AFP+CFP models are characterized by acyclicity of a binary relation inferred from the choice function and the model: for AFP model, $x'' \succ_{\text{AFP}} x'$ if there exist

	θ_{11}	θ_{21}	θ_{31}	θ_{41}	θ_{12}	θ_{22}	θ_{32}	θ_{42}	θ_{13}	θ_{23}	θ_{33}	θ_{43}	θ_{14}	θ_{24}	θ_{34}	θ_{44}	**b**
$\boldsymbol{\theta}_1^1$	-4	-4	-4	-4	-4	-4	-4	0	0	-4	-4	-4	-4	-4	-4	-4	0
$\boldsymbol{\theta}_2^1$	-4	-4	-4	-4	-4	-4	-4	1	0	-4	-4	-4	-4	-4	-4	-4	1
$\boldsymbol{\theta}_3^1$	-4	-4	-4	-4	-4	-4	-4	0	1	-4	-4	-4	-4	-4	-4	-4	1
$\boldsymbol{\theta}_4^1$	-4	-4	-4	-4	-4	-4	-4	0	0	-4	-4	-4	-4	-4	-4	-4	0
$\boldsymbol{\theta}_1^2$	-4	-4	-4	-4	-4	-4	-4	0	0	-4	-4	-4	-4	-4	-4	-4	0
$\boldsymbol{\theta}_2^2$	-4	-4	-4	-4	-4	-4	-4	0	0	-4	-4	-4	-4	-4	-4	-4	0
$\boldsymbol{\theta}_3^2$	-4	-4	-4	-4	-4	-4	-4	0	1	-4	-4	-4	-4	-4	-4	-4	1
$\boldsymbol{\theta}_4^2$	-4	-4	-4	-4	-4	-4	-4	0	0	-4	-4	-4	-4	-4	-4	-4	0
$\boldsymbol{\theta}_1^3$	-4	-4	-4	-4	-4	-4	-4	0	0	-4	-4	-4	-4	-4	-4	-4	0
$\boldsymbol{\theta}_2^3$	-4	-4	-4	-4	-4	-4	-4	0	0	-4	-4	-4	-4	-4	-4	-4	0
$\boldsymbol{\theta}_3^3$	-4	-4	-4	-4	-4	-4	-4	0	0	-4	-4	-4	-4	-4	-4	-4	0
$\boldsymbol{\theta}_4^3$	-4	-4	-4	-4	-4	-4	-4	0	0	-4	-4	-4	-4	-4	-4	-4	0

Table 8: Matrix Θ and vector b defined for RSM/TRSM test in Example 4.

$A, A' \subset X$ such that $x'' = f(A'), f(A') \neq f(A)$, and $A = A' \backslash x'$; for CFP model, $x'' \succ_{\text{CFP}} x'$ if there exist $A', A'' \subset X$ such that $f(A'') = x'', f(A') = x'$, and $\{x', x''\} \subset A'' \subset A'$; for AFP+CFP model, $x'' \succ_{\text{AFP+CFP}} x'$ if there exist $A, A', A'' \subset X$ such that $f(A'') = x'', f(A') = x', f(A') \neq f(A), A = A' \backslash x'$ and $\{x', x''\} \subset A'' \subset A'$. See Masatlioglu, Nakajima, and Ozbay (2012) for AFP, and Lleras, Masatlioglu, Nakajima, and Ozbay (2017) and its working paper version (2015) for CFP and AFP+CFP.

As shown in Manzini and Mariotti (2007), the choice function f is consistent with RSM model if and only if it satisfies,

- Weak WARP: for every $A, A', A'' \subset X$, $\{x', x''\} = A \subset A' \subset A''$ and $x'' = f(\{x', x''\}) = f(A'')$ implies $x' \neq f(A')$, and
- Expansion: for every $A, A', A'' \subset X$, $x = f(A') = f(A'')$ and $A = A' \cup A''$ implies $x = f(A)$.

Au and Kawai (2011) show that the choice function is consistent with TRSM model if and only if it satisfies Weak WARP, Expansion, and acyclicity of the following binary relation: $x'' \succ_{\text{TRSM}} x'$ if there exists $A', A'' \subset X$ such that $\{x', x''\} = A'' \subset A', x'' = f(A'')$, and $f(A') \neq f(A' \backslash x')$.

The above conditions are adapted to limited data environments as follows. Given a data set $\mathcal{O} = \{(a^t, A^t)\}_{t \in \mathcal{T}}$, for model $\mathbf{M} \in$ {AFP, CFP, AFP+CFP}, the binary relation $\succ_{\mathbf{M}}$ is defined in our context by rephrasing "there exists $A \subset X$" by "there exits $t \in \mathcal{T}$," and then we test acyclicity of this limited-data-based $\succ_{\mathbf{M}}$. For example, the binary relation in the AFP model is defined using a limited data set as follows: $x'' \succ_{\text{AFP}} x'$ if there exists $s, t \in \mathcal{T}$ such that $x'' = a^t, a^t \neq a^s$, and $A^s = A^t \backslash x'$. Similary, the conditions for testing RSM model can be molded into our context by rephrasing "for every $A \subset X$" by "for every $t \in \mathcal{T}$." For example, Weak WARP is expressed as: for every $r, s, t \in \mathcal{T}$, $\{x', x''\} = A^r \subset A^s \subset A^t$ and $x'' = a^r = a^t$ implies $x' \neq a^s$. The limited-data-based binary relation $\succ_{\text{TRSM}}$ of TRSM model is defined in a parallel fashion with AFP, CFP, and AFP+CFP models. Then we test TRSM by observing whether the data set obeys Weak WARP, Expansion, and acyclicity of this $\succ_{\text{TRSM}}$.

REMARK: For AFP, CFP, and AFP+CFP models, it is known that there are weak versions of WARP that characterize these limited consideration models. In particular, Masatligolu, Nakajima, and Ozbay (2012) show that there is an axiom WARP(LA) that is equivalent to acyclicity of $\succ_{\text{AFP}}$; Lleras, Masatlioglu, Nakajima, and Ozbay (2017) show that axiom WARP-CO is equivalent to the acyclicity of $\succ_{\text{CFP}}$; Lleras, Masatlioglu, Nakajima, and Ozbay (2015) show that axiom LC-WARP* is equivalent to the acyclicity of $\succ_{\text{AFP+CFP}}$. Since these equivalences break under a limited data set, we dealt only with the acyclicity conditions in testing these models.

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