DISCUSSION PAPER SERIES

Discussion paper No. 149

Limited consideration and limited data

Yuta Inoue
(Graduate School of Economics, Waseda University)

Koji Shirai
(School of Economics, Kwansei Gakuin University)

October 2016

SCHOOL OF ECONOMICS

KWANSEI GAKUIN UNIVERSITY

1-155 Uegahara Ichiban-cho
Nishinomiya 662-8501, Japan

Limited consideration and limited data

Yuta Inoue* and Koji Shirai†

October 15, 2016

Abstract

This paper develops revealed preference tests for choices under limited consideration, allowing a partially observed data set. Our tests cover leading theories in the literature including the limited attention model, the rationalization model, the categorize-then-choose model, and the rational shortlisting model. It is worth noting that all our tests exploit a common structure of limited consideration models. We start from a data set collected from a single agent, and then extend the analysis to panel data in which the coincidence of consideration sets/preferences of agents are tested.

KEYWORDS: Revealed preference; Limited consideration; Limited attention; Rational shortlisting; Categorization; Bounded rationality

JEL CLASSIFICATION NUMBERS: C6 D1 D8

* Graduate School of Economics, Waseda University. Email address: y.inoue@toki.waseda.jp
† School of Economics, Kwansei Gakuin University. Email address: kshirai1985@gmail.com

1 Introduction

Let X be a set that is interpreted as the set of alternatives, and let $A \subset X$ be a set of feasible alternatives for an agent. Following the classical choice theory, an agent will choose the most preferable alternative according to her preference which is often assumed to be complete, asymmetric, and transitive. In order to test if an agent's behavior can be accounted for by this standard framework, the theory of revealed preference is one of the most prevailing methods for economists. Typically, we collect finitely many observations of an agent's behavior $\mathcal{O} = \{(a^t, A^t)\}_{t \in \mathcal{T}}$, where $\mathcal{T}$ is the set of indices of observations, A^t is a set of feasible alternatives at observation t, and a^t is a chosen alternative from A^t. It is well known that a data set $\mathcal{O}$ is consistent with the standard choice framework, if and only if it obeys the *strong axiom of revealed preference (SARP)*, which requires acyclicity of the direct revealed preference relation $\succ^R$ defined as $x'' \succ^R x'$, if $x'' = a^t$ for some $t \in \mathcal{T}$, $x'' \neq x'$, and $x' \in A^t$.

However, as pointed out in a number of experimental studies (e.g. Tversky, 1969; Loomes, Starmer, and Sugden, 1991; and others), violation of SARP is not rare at all, and various theories of bounded rationality have been proposed for systematic analyses of cyclical choices. Amongst others, a number of studies investigate decision procedures where some feasible alternatives are a priori excluded from an agent's consideration. Namely, for a given feasible set A, an agent maximizes her preference relation not necessarily on A itself, but on some subset $\Gamma(A) \subset A$. For example, Lleras, Masatlioglu, Nakajima, and Ozbay (2010) and Masatlioglu, Nakajima, and Ozbay (2012) consider a situation where an agent is overwhelmed by the number of alternatives offered to her. In this case, due to the limitation of recognition capacity, she has to maximize her preference on a subset of the feasible set. As another example, Manzini and Mariotti (2007, 2012) and Cherepanov, Feddersen, and Sandroni (2013) establish shortlisting decision models. There, an agent has some criteria possibly different from her preference (e.g. psychological restrictions, a preference on categories rather than alternatives, and others), and she makes a sequential decision: an agent firstly makes a shortlist which is "optimal" in terms of her criteria, and then she chooses an alternative to maximize her preference relation. In this case, $\Gamma(A)$ can be interpreted as a shortlist derived in the first step.

As stated, by and large, the above listed bounded rationality models are inspired by observations of actual choice behavior. Then, it is natural to seek procedures for testing these models from agents' behavior. The principal objective of this paper is to develop such tests.

More specifically, for a data set $\mathcal{O} = \{(a^t, A^t)\}_{t\in\mathcal{T}}$, we provide a necessary and sufficient condition under which $\mathcal{O}$ is consistent with a model as follows: for every feasible set $A \subset X$, an agent maximizes some complete, asymmetric, and transitive preference $\succ$ on her *consideration set* $\Gamma(A) \subset A$. It is clear that, without any restriction on a set mapping Γ, such a model is vacuous in that any choice behavior is accounted for by letting $\{a^t\} = \Gamma(A^t)$ for every $t \in \mathcal{T}$. Thus, we deal with models where some restrictions *are* imposed on an agent's *consideration mapping* $\Gamma : 2^X \to 2^X$, which specifies her consideration set for every $A \subset X$. In particular, we start from looking at the following three restrictions: (1) the *attention filter property (AFP)*, which requires that for every $A \subset X$, $x \notin \Gamma(A) \Longrightarrow \Gamma(A\backslash x) = \Gamma(A)$, and (2) the *substitutable consideration (SUB)*, which requires that for every $A' \subset A''$, $\Gamma(A'') \cap A' \subset \Gamma(A')$, and (3) the *substitutable attention filter property (SAFP)*, which is the joint of AFP and SUB.[1] Loosely speaking, AFP requires that the removal of unrecognized alternatives does not change the set of recognized alternatives, while SUB requires that every alternative recognized at a larger feasible set must be recognized at a smaller feasible set.

A number of important decision procedures are covered by the above listed restrictions on a consideration mapping. First of all, the *limited attention* model in Masatlioglu, Nakajima, and Ozbay (2012) is nothing but a preference maximization model on a consideration mapping with AFP. In addition, we show that it is observationally equivalent to the *rational menu choice* model where an agent firstly chooses a menu according to an asymmetric and transitive menu preference on 2^X, and then chooses an alternative from the chosen menu. Second of all, the *order rationalization* model in Cherepanov, Feddersen, and Sandroni (2013) can be chracterized as a preference maximization model with a consideration mapping satisfying SUB. In addition, the *categorize-then-choose* model by Manzini and Mariotti (2012) also derives a consideration mapping that obeys SUB. We show that, as long as an agent has a complete, asymmetric, and transitive preference relation, the order rationalization model and the categorize-then-choose model are observationally equivalent.[2] If one admits that both AFP and SUB are reasonable restrictions on a consideration mapping, then it seems natural to require both of them, or SAFP on a consideration mapping. Indeed, as shown in Lleras, Masatlioglu, Nakajima and Ozbay (2010), many real-world examples actually support both AFP and SUB, or SAFP (e.g.

[1] Lleras et al. (2010) refer to SUB as the consideration filter property, and SAFP as the strong consideration filter property.

[2] In the original setting in Manzini and Mariotti (2012), an agent's preference is assumed to be just complete and asymmetric.

paying attention to n most advertised commodities). As easily deduced, if a menu preference obeys substitutablity, then a rational menu choice model is observationally equivalent to a substitutable limited attention model.

What is not covered by the above three types of restrictions is the *rational shortlisting* model in Manzini and Mariotti (2007). There, an agent makes a shortlist as the set of maximal elements of an asymmetric first step preference, and then she makes a choice to maximize her preference relation. Regarding a shortlist as a consideration set, a consideration mapping must obey SUB, but it has stronger observable restrictions. Indeed, even if a data set is rationalizable by a limited consideration model with SAFP, it may not be supported as a result of a rational shortlisting model, which is shown in Lleras, Masatlioglu, Nakajima, and Ozbay (2010) in the framework of full-observation. In addition, under the *transitive rational shortlisting* model where a first step preference is asymmetric and transitive, a consideration mapping obeys SAFP, but again, such a model has stronger observable restrictions than a limited consideration model with SAFP.[3] In order to cover these models, we provide follow-up tests that can be applied to data that are consistent with SUB/SAFP limited consideration models.

While limited consideration models are developed in the framework of single agent decision models, they also invoke some issues on multi-agent contexts. Due to the nature of limited consideration models, the source of different choices between agents can be disentangled into the following two components; one is the difference of preferences and the other is the difference of consideration mappings. In other words, even if the choices made by agents are diverse, they may have common preferences or consideration mappings. To distinguish the source of variety of choices may be of importance in some contexts. In applied studies, it is often assumed that agents in a population can be partitioned into several types, and that agents with the same type share some common behavioral procedure. It is commonly assumed that agents in the same type have the same preference. In addition, if a consideration mapping reflects some psychological restrictions or social norms, then the coincidence of a consideration mapping within a group also seems plausible. Testing these hypotheses does not require too much: our revealed preference analyses can easily be extended to the case of panel data in the form of $\mathcal{O} = \{((a_i^t)_{i\in N}, A^t)\}_{t\in\mathcal{T}}$, where N being a set of agents.

[3] Similar to the case of the categorize-then-choose model, the original setting in Manzini and Mariotti (2007) does not require the transitivity, while Au and Kawai (2011), which firstly investigates the transitive rational shortlisting model, does require the transitivity also on a second step preference relation.

From a technical viewpoint, all our revealed preference tests are based on a common insight as follows. Note that, if a data set $\mathcal{O} = \{(a^t, A^t)\}_{t\in\mathcal{T}}$ obeys SARP, or it contains no cycle with respect to $\succ^R$, then such a data set is trivially consistent with a limited consideration model by letting $\Gamma(A) = A$ for every $A \subset X$.[4] Now, suppose that a data set is consistent with some limited consideration model, but *does* have a cycle $a^{t_1} \succ^R a^{t_2} \succ^R \cdots \succ^R a^{t_K} \succ^R a^{t_1}$. As long as an agent's preference is complete, asymmetric, and transitive, there exists at least one k such that $a^{t_{k+1}} \succ a^{t_k}$ despite $a^{t_k} \succ^R a^{t_{k+1}}$. We refer to such a^{t_k} as a *cut-off point* of a cycle. Then, it requires that $a^{t_{k+1}} \notin \Gamma(A^{t_k})$, and the specific property of Γ derived from the model of interest, in turn, delivers further restrictions. For example, if we are testing the substitutable consideration model, then SUB requires that for every $s \in \mathcal{T}$ such that $A^{t_k} \subset A^s$, $\Gamma(A^s) \not\ni a^{t_{k+1}}$, and hence, $a^s \neq a^{t_{k+1}}$ must hold. Given this argument, loosely speaking, the consistency of a data set with a specific choice model hinges on the existence of cut-off points that do not cause any contradiction with the model. As we will see in Section 5, this method remains valid even for the case of panel data.

Lastly, we briefly relate our results with the existing literature. It is standard in the literature of bounded rationality that the observable restrictions are investigated using an exhaustive data set, or a choice function. For example, Lleras, Masatlioglu, Nakajima, and Ozbay (2010) and Masatlioglu, Nakajima, and Ozbay (2012) characterize AFP, SUB, SAFP, and the transitive rational shortlisting in terms of a restriction on a choice function. However, these results are not extendable to partially observed data sets. The most closely related papers are Tyson (2013) and de Clippel and Rozen (2014), both of which are pioneering works of testing limited consideration models with partially observed data.[5] Their approaches are different from ours and even between themselves. Roughly speaking, their revealed preference tests nicely work for SUB, but have some difficulties for AFP. In de Clippel and Rozen's approach, a necessary and sufficient condition is stated in terms of the existence of a specific type of binary relation, where some restrictions are imposed on the relations among unobserved alternatives as well as the ones among observed choices. They provide a procedure for finding out such a binary relation, however, regarding the limited attention model, it works only for a specific type of data. On the other hand, loosely speaking, our revealed preference tests complete "within" observed actions, and are not affected by the strucutre of a data set. The

[4]For example, it is clear that such a consideration mapping obeys SAFP.

[5]It should be also noted that, to the best of the authors' knowledge, de Clippel and Rozen (2014) is the first work that explicitly points out the above stated non-extendability problem.

decision model with SAFP consideration, rational shortlisting type models, and panel data are not dealt with in Tyson (2013) and de Clippel and Rozen (2014).

The rest of this paper is organized as follows. In Section 2, we introduce limited consideration models that are dealt with in this paper. Some observational equivalence between models are also provided. The revealed preference tests for limited consideration models with AFP, SUB, and SAFP are developed in Section 3, and tests for rational shortlisting type models are stated in Section 4. We extend the tests to the case of panel data in Section 5.

2 Choices under limited consideration

Consider a single agent decision problem where X is a finite set of alternatives, and $\succ$ is a complete, asymmetric, and transitive preference of an agent, which we refer to as a strict preference.[6] If an agent obeys the rational choice model, then for every feasible set $A \subset X$, she maximizes her strict preference $\succ$ on A.

On the other hand, motivated by evidences contradicting the rational choice theory, a number of alternative decision procedures are proposed in the literature of bounded rationality. There, either consciously or unconsciously, an agent makes a shortlist of alternatives before she chooses an alternative. That is, there exists a *consideration mapping* $\Gamma : 2^X \to 2^X$ such that $\Gamma(A) \subset A$ for every $A \subset X$, and an agent maximizes her strict preference on $\Gamma(A)$, rather than A itself. In what follows, given a consideration mapping Γ, $\Gamma(A)$ is referred to as a *consideration set* on A. Furthermore, in general, we refer to a pair of a consideration mapping and a strict preference $(\Gamma, \succ)$ as a *limited consideration* model.

2.1 Models with AFP

In Masatlioglu, Nakajima, and Ozbay (2012), they consider a situation in which an agent cannot recognize all feasible alternatives due to the limitation of recognition capacity. There, following psychological literature, a consideration mapping Γ is supposed to have the *attention filter property (AFP)* defined as; for every $A \subset X$ and $x \in A$,

$$x \notin \Gamma(A) \Longrightarrow \Gamma(A \backslash x) = \Gamma(A). \tag{1}$$

[6]For every $x \in X$, $x \not\succ x$, and for every distinct $x, y \in X$, either $x \succ y$ or $y \succ x$ holds, and for every distinct $x, y, z \in X$, $x \succ y$ and $y \succ z$ imply $x \succ z$.

This implies that the consideration set is not affected when unrecognized elements are removed from a feasible set. Alternatively, (1) is rewritten as; for every $A \subset X$ and $B \subset A$,

$$\Gamma(A) \subset A \backslash B \Longrightarrow \Gamma(A \backslash B) = \Gamma(A). \tag{2}$$

In what follows, we refer to $(\Gamma, \succ)$ as a *limited attention* model, if Γ obeys AFP.

In fact, this type of consideration mapping can endogenously be derived from a conscious choice of menus from a feasible set. Consider an asymmetric and transitive binary relation $\succ^M$ defined on 2^X, to which we refer as a *menu preference*. Let for every $A \subset X$,

$$\Gamma(A) = \{x \in A : \exists B \ni x \text{ such that } B' \nsucc^M B \text{ for every } B' \subset A\}, \tag{3}$$

which is saying that $x \in \Gamma(A)$ if it is contained in a maximal menu $B \subset A$ with respect to $\succ^M$. We refer to $(\Gamma, \succ)$ as a *rational menu choice* model, if Γ can be written in the form of (3) for some asymmetric and transitive menu preference $\succ^M$. The motivation of this terminology seems clear: an agent firstly chooses some menu from feasible alternatives, and then maximizes her strict preference $\succ$ on the chosen menu. It should also be noted that, in rational menu choice models, no particular connection between $\succ^M$ and $\succ$ is required. Hence, for example, this model can cover a situation in which a menu and an alternative are respectively chosen by different agents (e.g. the mother may a priori restrict the set of toys from which the child chooses). As a slightly more restrictive version, we refer to $(\Gamma, \succ)$ as a *complete rational menu choice* model, if Γ can be written as (3) for some complete, asymmetric, and transitive menu preference $\succ^M$.

It turns out that limited attention models, rational menu choice models, and complete rational menu choice models are all observationally equivalent in the sense that if an agent's behavior is accounted for by one of them, then it is also consistent with others. This result has not been shown in the literature.

PROPOSITION 1. *Limited attention models, rational menu choice models, and complete rational menu choice models are all observationally equivalent.*

2.2 Models with SUB

As an alternative structure of a consideration mapping, Lleras, Masatlioglu, Nakajima, and Ozbay (2010) consider the following restriction: for every $A' \subset A''$ and $x \in A'$,

$$x \notin \Gamma(A') \Longrightarrow x \notin \Gamma(A''). \tag{4}$$

In words, if an alternative is not recognized in a smaller feasible set, then it cannot be recognized in a larger feasible set. This seems plausible if an agent has limited capacity of recognition. Equivalently, (4) can be written as; for every $A' \subset A''$,

$$\Gamma(A'') \cap A' \subset \Gamma(A'). \tag{5}$$

This condition is nothing but the substitutability often used in the literature of matching theory, which is equivalent to the monotonicity of the set of unrecognized alternatives. We say that Γ obeys *substitutability (SUB)* if it obeys (5), and $(\Gamma, \succ)$ is referred to as a *substitutable consideration* model, if Γ obeys SUB.

Similar to the case of limited attention models, a substitutable consideration mapping can be generated by conscious shortlisting. In Cherepanov, Feddersen, and Sandroni (2013), they consider a situation in which an agent has some criteria on alternatives, other than her strict preference. Each criterion is referred to as a *rationale*, which may be a psychological restriction or may be a social norm. A set of rationales of an agent is denoted by $\{R^k\}_{k=1}^K$, each of which is assumed to be just a binary relation, so it may not be complete, asymmetric, or transitive. An alternative $x \in X$ is said to be supported on $A \subset X$, if there exists some rationale R^k such that xR^kx' for all $x' \in A \backslash x$. Then, an agent is supposed to eliminate all unsupported alternatives from a feasible set, that is, a consideration mapping is defined such that for every $A \subset X$,

$$\Gamma(A) = \{x \in A : \exists R^k \text{ s.t. } xR^kx' \text{ for all } x' \in A \backslash x\}. \tag{6}$$

We refer to $(\Gamma, \succ)$ as an *order rationalization* model, if Γ is represented as (6) for some set of rationales $\{R^k\}_{k=1}^K$. It is not difficult to see that a consideration mapping in (6) obeys SUB, and as shown in Cherepanov, Feddersen, and Sandroni (2013), the converse is also true.

Another observationally equivalent decision model is a *categorize-then-choose* model in

Manzini and Mariotti (2012). In their model, an agent has a *shading relation* $\succ^M$, which is assumed to be asymmetric on 2^X. In the first step, an agent makes a shortlist such that for every $A \subset X$,

$$\Gamma(A) = \{x \in A : \nexists B', B'' \subset A \text{ such that } B'' \succ^M B' \text{ and } x \in B'\}. \tag{7}$$

Loosely speaking, an alternative in a dominated category is eliminated from candidates of her choice, and then, in the second step, an agent maximizes her strict preference $\succ$ on $\Gamma(A)$. We say that $(\Gamma, \succ)$ is a categorize-then-choose model, if Γ is represented by (7) for some asymmetric shading relation $\succ^M$ on 2^X. Again, it is not difficult to see that $\Gamma(A)$ obeys SUB, though it is formally proved in Appendix. Moreover, we have the following observational equivalence.

PROPOSITION 2. *Substitutable consideration models, order rationalization models, and categorize-then-choose models are all observationally equivalent.*

In Cherepanov, Feddersen, and Sandroni (2013) and Manzini and Mariotti (2012), they also deal with models with weaker restrictions, both of which are shown to be equivalent to WWARP. When the completeness and transitivity is dropped from $\succ$ (i.e. preference is just asymmetric), then the decision procedure is referred to as a *basic rationalization model*, which is equivalent to WWARP. On the other hand, in Manzini and Mariotti (2012), they showed that a categorize-then-choose model is equivalent to WWARP, if a preference relation is complete and asymmetric. Since we have assumed that preference is complete, asymmetric and transitive, strictly speaking, the observational equivalence in Proposition 2 is not explicitly stated in the literature.

2.3 Models with SAFP

If we admit that both AFP and SUB are respectively reasonable, then it is natural to consider the joint of AFP and SUB. Indeed, as pointed out in Lleras, Masatlioglu, Nakajima, and Ozbay (2010), both AFP and SUB are plausible in a number of real-world examples. For example, consider the situations in which an agent pays attention to; (a) n-most advertised commodities, (b) all commodities of a specific brand, and if there are none available, then all commodities of another specific brand, and (c) n-top candidates in each field in job markets. All of these decision procedures derive consideration mappings satisfying both AFP and SUB.

We say that Γ obeys the *substitutable attention filter property (SAFP)*, if it obeys both AFP and SUB, and a pair $(\Gamma, \succ)$ is referred to as a *substitutable limited attention* model, if Γ obeys SAFP.

Given Proposition 1, it is straightforward that substitutable limited attention models are observationally equivalent to rational menu choice models with a substitutable menu preference. Here, similar to the case of matching theory, a menu preference $\succ^M$ on 2^X is substitutable, if its maximal set function obeys SUB. We refer to $(\Gamma, \succ)$ as a *(complete) substitutable rational menu choice* model, if Γ can be written as (3) with $\succ^M$ being substitutable (and complete).

PROPOSITION 3. *Substitutable limited attention models, substitutable rational menu choice models, and substitutable complete rational menu choice models are all observationally equivalent.*

2.4 Rational shortlisting

Substitutable limited attention models can be related to Manzini and Mariotti (2007)'s two-step decision procedure called a *rational shortlisting* model. There, an agent has a preference relation for each step, say $\succ'$ and $\succ$, and for every $A \subset X$, an agent firstly makes a shortlist $\Gamma(A)$ such that

$$\Gamma(A) = \{x \in A : \nexists x' \text{ such that } x' \succ' x\}, \tag{8}$$

and then, in the second step, an agent maximizes her second step preference relation $\succ$ on $\Gamma(A)$. In Manzini and Mariotti (2007), the first step preference $\succ'$ is just assumed to be acyclic, while Au and Kawai (2011) deal with the case where $\succ'$ is asymmetric and tansitive.[7] We refer to $(\Gamma, \succ)$ as a *(transitive) rational shortlisting* model, if Γ can be represented in the form of (8) for some acyclic (asymmetric and transitive) binary relation $\succ'$. Note that, similar to other models, we require that the second step preference $\succ$ is a strict preference. The following statement is known in the literature, but we record it for future references.

[7]In Manzini and Mariotti (2007), they assumed that both $\succ'$ and $\succ$ are just asymmetric. However, since they also assume that the choice function is nonempty for all $A \subset X$, it is clear that $\succ'$ must be acyclic (otherwise $\Gamma(A)$ would be empty for some A).

PROPOSITION 4. *If an agent obeys a (transitive) rational shortlisting model, then she obeys a substitutable consideration (substitutable limited attention) model.*

The proof of the proposition is almost obvious. If Γ is defined as (4) for some acyclic binary relation $\succ'$ and $A' \subset A''$, then $x \notin \Gamma(A')$ implies the existence of some $y \in A'$ such that $y \succ' x$. Then, $y \in A' \subset A''$ implies that $x \notin \Gamma(A'')$. The transitive rational shortlisting part can be confirmed in a similar vein.

3 Testing AFP, SUB, and SAFP

It is well known that the rational choice theory can be easily tested from agent's observed choice behavior. Let $\mathcal{O} = \{(a^t, A^t)\}_{t \in \mathcal{T}}$ be a finite set of observed choices, where $\mathcal{T} = \{1, 2, ..., T\}$ is a set of indices of observations, $A^t \subset X$ be the feasible set at observation t, and $a^t \in A^t$ be the chosen alternative at $t \in \mathcal{T}$. A key for testing the rational choice model is the *direct revealed preference* relation $\succ^R$ defined as $x'' \succ^R x'$, if $x'' = a^t$ for some $t \in \mathcal{T}$, $x'' \neq x'$, and $x' \in A^t$. In the case of the rational choice theory, the motivation of this terminology is obvious. Indeed, if an agent follows the rational choice model and $x'' \succ^R x'$ for some $x'', x' \in X$, then $\succ^R$ must be contained in the agent's "true" preference $\succ$, and hence, $\succ^R$ cannot have a cycle, that is,

$$a^{t_1} \succ^R a^{t_2} \succ^R \cdots \succ^R a^{t_K} \Longrightarrow a^{t_K} \not\succ^R a^{t_1}. \tag{9}$$

Actually, the acyclicity of $\succ^R$, which is referred to as the *strong axiom of revealed preference (SARP)*, fully characterizes the observable restrictions from the rational choice model.

The objective of this section is to develop counterparts of SARP for testing limited consideration models with AFP, SUB, and SAFP. Given a data set $\mathcal{O} = \{(a^t, A^t)\}_{t \in \mathcal{T}}$, we say that a data set is *rationalizable* by a specific limited consideration model $(\Gamma, \succ)$, if for every $t \in \mathcal{T}$, $a^t \in \Gamma(A^t)$ and $a^t \succ x$ for all $x \in \Gamma(A^t) \backslash a^t$. Clearly, without any restriction on Γ, we can trivially rationalize any observed choices by letting for every $t \in \mathcal{T}$, $\Gamma(A^t) = \{a^t\}$. The restrictions AFP, SUB, and SAFP respectively exclude this trivial rationalization, but they still allow the possibility of cyclical choices, i.e. a data set $\mathcal{O} = \{(a^t, A^t)\}_{t \in \mathcal{T}}$ may contain cycles with respect to $\succ^R$. Naturally, as described below, investigating the structure of cycles plays a key role in testing limited consideration models. It is also worth noting that Theorems 1, 2, and 3 below do not depend on the finiteness of X.

3.1 A general principle

Before proceeding to our main results, we now put forward a general idea for testing limited consideration models. In testing a specific limited consideration model, we are actually testing two hypotheses simultaneously: one is the structure of a consideration mapping required in the model, and the other is that an agent has a strict preference. In other words, when we say that a data set is consistent with some limited consideration model, we have to find a pair $(\Gamma, \succ)$ that does not contradict observed choices. As already mentioned, if a data set $\mathcal{O} = \{(a^t, A^t)\}_{t\in\mathcal{T}}$ does not have a cycle with respect to $\succ^R$, then, it is trivially supported as a result of a limited consideration model with Γ being an identity mapping. Thus, with no loss of generality, we may concentrate on a situation where $\mathcal{O}$ does have cycles. Formally, a profile of chosen alternatives $(a^{t_k})_{k=1}^K$ is a *cycle* with respect to $\succ^R$, if for every $k \leqslant K$, $a^{t_k} \succ^R a^{t_{k+1}}$ and $a^{t_K} = a^{t_1}$. A cycle $(a^{t_k})_{k=1}^K$is *minimal*, if it contains no cycle other than itself. We assume that a data set $\mathcal{O}$ has $Q(\geqslant 0)$-minimal cycles with respect to $\succ^R$, and for $q \geqslant 1$, the q-th minimal cycle is represented as $(a^{t_k^q})_{k=1}^{K_q}$. In what follows, for simplicity, we refer to a minimal cycle as a cycle.

Suppose that a data set $\mathcal{O} = \{(a^t, A^t)\}_{t\in\mathcal{T}}$ is collected from an agent obeying a specific limited consideration model. Since we assume that an agent's preference $\succ$ is asymmetric and transitive, for every cycle $(a^{t_k^q})_{k=1}^{K_q}$, there exists at least one $k(q)$ such that $a^{t^q_{k(q)}} \succ^R a^{t^q_{k(q)+1}}$, but $a^{t^q_{k(q)+1}} \succ a^{t^q_{k(q)}}$. We refer to such an $a^{t^q_{k(q)}}$ as a *cut-off point* of a cycle $(a^{t_k^q})_{k=1}^{K_q}$. Let us denote such cut-off point by $a^{t(q)}$, and the alternative that succeeds it in the cycle by $b^{t(q)}$, i.e. $a^{t(q)} := a^{t^q_{k(q)}}$ and $b^{t(q)} := a^{t^q_{k(q)+1}}$. An agent is supposed to maximize her preference on $\Gamma(A)$ for every $A \subset X$, and hence, $b^{t(q)} \succ a^{t(q)}$ implies that $b^{t(q)} \in A^{t(q)}$ but $b^{t(q)} \notin \Gamma(A^{t(q)})$. Moreover, if there exist $q, q' \in \{1, \ldots, Q\}$ such that $a^{t(q)} = a^{t(q')}$, i.e. the cut-off points correspond to the same alternative for cycles q and q', it also holds that $b^{t(q')} \notin \Gamma(A^{t(q)})$. To generalize the argument, define for every $t \in \mathcal{T}$,

$$B^t = \{b^{t(q)} \in A^t : a^{t(q)} = a^t\}, \tag{10}$$

which is empty if there exists no cut-off point $a^{t(q)}$ such that $a^t = a^{t(q)}$. This set B^t plays crucial roles in our revealed preference tests. Since $x \succ a^t$ holds for every $x \in B^t$, we have that

$x \notin \Gamma(A^t)$. Put otherwise, for every $t \in \mathcal{T}$,

$$\Gamma(A^t) \subset A^t \backslash B^t. \tag{11}$$

Note that the constraint on Γ in (11) does not depend on the type of model. That is, as long as an agent obeys some limited consideration model, her consideration mapping must satisfy it. On the other hand, since the set B^t depends on cut-off points of cycles, some restriction must be imposed in order for Γ to satisfy a specific structure like AFP, SUB, or SAFP. Besides, if a data set is consistent with some limited consideration model $(\Gamma, \succ)$, it must also hold that for every $t \in \mathcal{T}$, $a^t \in \Gamma(A^t)$, which, in view of (11), also invokes restrictions on properties of cut-off points.

In essence, whether a data set can be rationalized by a specific model $(\Gamma, \succ)$ hinges on whether an observer can choose a cut-off point $a^{t(q)}$ from every cycle $(a^{t^q_k})_{k=1}^{K_q}$ so that it is consistent with (i) the properties of Γ required in that model, and (ii) preference maximizing behavior on $\Gamma(A^t)$. In the following subsections, revealed preference tests for the limited attention, the substitutable consideration, and the substitutable limited attention are shown in order.

3.2 Attention filter property

Suppose that $\mathcal{O} = \{(a^t, A^t)\}_{t \in \mathcal{T}}$ is a data set collected from an agent obeying a limited attention model $(\Gamma, \succ)$, i.e. Γ obeys AFP defined in (2). With no loss of generality, we may assume that $\mathcal{O}$ contains cycles with respect to $\succ^R$, and it is clear from the discussion in the previous subsection that each cycle has at least one cut-off point. Let $\left[a^{t(q)}\right]_{q=1}^{Q}$ be a profile of cut-off points out of Q cycles (one cut-off point is chosen from each cycle). In addition to (11), the limited attention model casts further restrictions on the cut-off points and the value of a consideration mapping. Given (11), since Γ must obey AFP, it holds that, for every $t \in \mathcal{T}$,

$$(A^t \backslash B^t) \subset A \subset A^t \Longrightarrow \Gamma(A) = \Gamma(A^t). \tag{12}$$

The above derives the following important restriction. Given a profile of cut-off points $\left[a^{t(q)}\right]_{q=1}^{Q}$, suppose that $(A^s \backslash B^s) \cup (A^t \backslash B^t) \subset (A^s \cap A^t)$ hold. Then, by letting $A = (A^s \backslash B^s) \cup (A^t \backslash B^t)$, the LHS of (12) is satisfied both for s and t. As a result, it must hold that $\Gamma(A^s) =$

$\Gamma((A^s \backslash B^s) \cup (A^t \backslash B^t)) = \Gamma(A^t)$, which implies that $a^s = a^t$. In fact, this property, which is summarized as the axiom below, characterizes a data set that is rationalizable by the limited attention model:

Axiom of Limited Attention (ALA): A data set $\mathcal{O} = \{(a^t, A^t)\}_{t \in \mathcal{T}}$ obeys the axiom of limited attention, if there exists a profile of cut-off points $\left[a^{t(q)}\right]_{q=1}^{Q}$ such that for every $t \in \mathcal{T}$,

$$(A^s \backslash B^s) \cup \left(A^t \backslash B^t\right) \subset (A^s \cap A^t) \Longrightarrow a^s = a^t. \tag{13}$$

Recall that, as seen from its definition in (10), the set B^t depends on the choice of a profile of cut-off points. It is already clear that ALA is necessary for a data set $\mathcal{O}$ to be rationalizable by a limited consideration model, but our more substantial claim in the theorem is the converse: if a data set $\mathcal{O}$ obeys ALA, then an agent's behavior can be accounted for by the limited attention model.

Theorem 1. *A data set $\mathcal{O} = \{(a^t, A^t)\}_{t \in \mathcal{T}}$ is rationalizable by a limited attention model, if and only if it obeys ALA.*

Our proof for the sufficiency part is constructive. Given a profile of cut-off points that obeys (13), we explicitly construct a consideration mapping that obeys AFP. Then by using it, a strict preference that rationalizes a data set is also constructed. Specifically, given a profile of cut-off points satisfying (13), we simply define a consideration mapping Γ such that for every $A \subset X$,

$$\Gamma(A) = A \backslash B^t \text{ for } t \in \mathcal{T} \text{ such that } A^t \backslash B^t \subset A \subset A^t. \tag{14}$$

In general, for a given $A \subset X$, there may be multiple observations that satisfy the condition in (14), i.e. for some $s, t \in \mathcal{T}$, $A^t \backslash B^t \subset A \subset A^t$ and $A^s \backslash B^s \subset A \subset A^s$. However, in that case, $A \backslash B^t = A \backslash B^s$ must hold, and hence, the above construction of Γ is well-defined, which is proved in Appendix.

Lemma 1. *Suppose that for some $s, t \in \mathcal{T}$, $A^t \backslash B^t \subset A \subset A^t$ and $A^s \backslash B^s \subset A \subset A^s$. Then, it holds that $A \backslash B^t = A \backslash B^s$.*

Based on Γ defined as (14), the proof essentially completes with the help of the following two lemmas that are proved in Appendix.

LEMMA 2. *The consideration mapping* Γ *defined as (14) obeys AFP.*

LEMMA 3. *Let* $\succ^*$ *be a binary relation such that* $x'' \succ^* x'$, *if* $x'' = a^t$ *for some* $t \in \mathcal{T}$, $x' \in \Gamma(A^t)$, *and* $x'' \neq x'$. *Then,* $\succ^*$ *is acyclic and for every* $t \in \mathcal{T}$, $a^t \in \Gamma(A^t)$.

The rest of the proof is somewhat routine work: by Lemma 3, the transitive closure of $\succ^*$ is an asymmetric and transitive ordering, and hence, by Szpilrajn's theorem, it can be extended to a strict preference $\succ$ on X. In addition, again by Lemma 3, it holds that for every $t \in \mathcal{T}$, $a^t \succ x$ for every $x \in \Gamma(A^t) \backslash a^t$. Then, together with Lemma 2, the data set is rationalizable by the limited attention model $(\Gamma, \succ)$.

Lastly, gathering together with Proposition 1, Theorem 1 has the following corollary.

COROLLARY 1. *The following statements are all equivalent.*

(a) A data set $\mathcal{O} = \{(a^t, A^t)\}_{t \in \mathcal{T}}$ *obeys ALA.*

(b) A data set $\mathcal{O} = \{(a^t, A^t)\}_{t \in \mathcal{T}}$ *is rationalizable by a limited attention model.*

(c) A data set $\mathcal{O} = \{(a^t, A^t)\}_{t \in \mathcal{T}}$ *is rationalizable by a rational menu choice model.*

(d) A data set $\mathcal{O} = \{(a^t, A^t)\}_{t \in \mathcal{T}}$ *is rationalizable by a complete rational menu choice model.*

3.3 Substitutable consideration

The issue in this subsection is to develop a revealed preference test for a limited consideration model $(\Gamma, \succ)$ where Γ obeys SUB. Suppose that $\mathcal{O} = \{(a^t, A^t)\}_{t \in \mathcal{T}}$ is collected from an agent obeying a substitutable consideration model. Again, without loss of generality, we may assume that $\mathcal{O}$ contains cycles with respect to $\succ^R$ and each of them has at least one cut-off point. By letting $\left[a^{t(q)}\right]_{q=1}^{Q}$ be a profile of cut-off points, we have (11). Bearing this in mind, consider any $s, t \in \mathcal{T}$ such that $A^s \subset A^t$. Then, considering SUB defined in (5), it must hold that $\Gamma(A^t) \cap A^s \subset \Gamma(A^s)$. In addition, by (11), this implies that $\Gamma(A^t) \cap B^s = \varnothing$, which in turn implies that $a^t \notin B^s$. In fact, this simple observation completely characterizes whether a data set is consistent with the substitutable consideration model.

AXIOM OF SUBSTITUTABLE CONSIDERATION (ASC): A data set $\mathcal{O} = \{(a^t, A^t)\}_{t \in \mathcal{T}}$ obeys the axiom of substitutable consideration, if there exists a profile of cut-off points $\left[a^{t(q)}\right]_{q=1}^{Q}$ such that for every $s, t \in \mathcal{T}$,

$$A^s \subset A^t \Longrightarrow a^t \notin B^s.[8] \tag{15}$$

[8] Once again, the set B^s in the axiom depends on the choice of a profile of cut-off points.

Theorem 2. *A data set $\mathcal{O} = \{(a^t, A^t)\}_{t\in\mathcal{T}}$ is rationalizable by a substitutable consideration model, if and only if it obeys ASC.*

The proof of Theorem 2 is parallel to that of Theorem 1. The necessity of ASC has already been discussed, and the proof for sufficiency is constructive. First of all, given a profile of cut-off points $\left[a^{t(q)}\right]_{q=1}^{Q}$, we define a consideration mapping Γ such that for every $A \subset X$,

$$\Gamma(A) = A \Big\backslash \bigcup_{t:A^t \subset A} B^t. \tag{16}$$

The substantial parts of the proof are to show that Γ constructed as above obeys SUB, and that the binary relation $\succ^*$ defined as $x'' \succ^* x'$ if $x'' = a^t$, $x' \in \Gamma(A^t)$, and $x'' \neq x'$ is acyclic, which are proved in Appendix.

Lemma 4. *The consideration mapping defined as (16) obeys SUB.*

Lemma 5. *Let $\succ^*$ be a binary relation such that $x'' \succ^* x'$, if $x'' = a^t$ for some $t \in \mathcal{T}$, $x' \in \Gamma(A^t)$, and $x'' \neq x'$. Then, $\succ^*$ is acyclic and for every $t \in \mathcal{T}$, $a^t \in \Gamma(A^t)$.*

The rest of the proof is again similar to the case of Theorem 1, just extending the transitive closure of $\succ^*$ to a strict preference $\succ$ by using Szpilrajn's theorem, which is easily proved to rationalize a data set by the substitutable consideration model $(\Gamma, \succ)$.

Given Proposition 2, Theorem 2 has the following corollary to which a similar statement can be found in de Clippel and Rozen (2014).

Corollary 2. *The following statements are all equivalent.*

(a) A data set $\mathcal{O} = \{(a^t, A^t)\}_{t\in\mathcal{T}}$ obeys ASC.

(b) A data set $\mathcal{O} = \{(a^t, A^t)\}_{t\in\mathcal{T}}$ is rationalizable by a substitutable consideration model.

(c) A data set $\mathcal{O} = \{(a^t, A^t)\}_{t\in\mathcal{T}}$ is rationalizable by an order rationalization model.

(d) A data set $\mathcal{O} = \{(a^t, A^t)\}_{t\in\mathcal{T}}$ is rationalizable by a categorize-then-choose model.

3.4 Substitutable limited attention

In the rest of this section, we deal with a revealed preference characterization of substitutable limited attention models. Clearly, if a data set $\mathcal{O} = \{(a^t, A^t)\}_{t\in\mathcal{T}}$ is rationalizable by a substitutable limited attention model, then it is also consistent with both limited attention and substitutable consideration. Hence, by Theorems 1 and 2, such a data set must obey both AFP and SUB. However, as we shall show in Example 1 below, the joint of ALA and ASC is insufficient to characterize the observable restrictions of such models.

To clarify a necessary condition, let $\mathcal{O} = \{(a^t, A^t)\}_{t\in\mathcal{T}}$ be a data set collected from an agent obeying a substitutable limited attention model. That is, an agent has a strict preference $\succ$ on X and a consideration mapping Γ that obeys SAFP. Similar to the previous cases, we may assume that $\mathcal{O}$ contains cycles with respect to $\succ^R$, and let $\left[a^{t(q)}\right]_{q=1}^{Q}$ be a profile of cut-off points. Corresponding to this profile of cut-off points, the set B^t is determined as in (10) for every $t \in \mathcal{T}$. Since $\mathcal{O}$ must obey AFP, together with (11), it holds that for every $t \in \mathcal{T}$, $\Gamma(A^t) = \Gamma(A^t\backslash B^t)$. In particular, $\Gamma(A^t) \subset A^t\backslash B^t$ must hold. In fact, as a slight extenstion of this, for $s, t \in \mathcal{T}$ it holds that

$$(A^s\backslash B^s) \subset A^t \Longrightarrow \Gamma(A^t) \subset A^t\backslash B^s. \tag{17}$$

To see (17), we employ both AFP and SUB. First, notice that if $(A^s\backslash B^s) \subset A^t$ holds, then there exist some sets $C \subset B^s$ and $D \subset X\backslash A^s$ such that $A^t = [(A^s\backslash B^s) \cup C \cup D]$.[9] Since, obviously, $(A^t \cap A^s) = [(A^s\backslash B^s) \cup C]$, it holds that $(A^s\backslash B^s) \subset (A^t \cap A^s) \subset A^s$. Then, gathering together with $\Gamma(A^s) = \Gamma(A^s\backslash B^s)$, AFP implies that $\Gamma(A^t \cap A^s) = \Gamma(A^s\backslash B^s)$. In addition, since $\left[\Gamma(A^t) \cap (A^t \cap A^s)\right] = (\Gamma(A^t) \cap A^s)$, SUB implies that $(\Gamma(A^t) \cap A^s) \subset \Gamma(A^t \cap A^s)$. Gathering together with $\Gamma(A^t \cap A^s) = \Gamma(A^s\backslash B^s)$, this implies that $(\Gamma(A^t) \cap A^s) \subset A^s\backslash B^s$. Since $B^s \subset A^s$, we conclude that $\Gamma(A^t) \cap B^s = \varnothing$. As long as an agent obeys a substitutable limited attention model, it must hold that $a^t \in \Gamma(A^t)$, and the relationship (17) impose a restriction on the relationship between chosen elements and cut-off points (or B^t's corresponding to them):

$$(A^s\backslash B^s) \subset A^t \Longrightarrow a^t \notin B^s. \tag{18}$$

As a matter of fact, the conclusions in (17) and (18) have further room for extension, which plays a key role in characterizing substitutable limited attention. We start from extending (17). Looking at the argument in the preceding paragraph, one can see that the facts of $\Gamma(A^s) \subset A^s\backslash B^s$ and $B^s \subset A^s$ are cornerstones, and once they are known, (17) follows from AFP and SUB. That is, even for general subsets $A', A'' \subset X$, if both $\Gamma(A') \subset A'\backslash V$ and $A'\backslash V \subset A''$ hold for some $V \subset A'$, then $\Gamma(A'') \subset A''\backslash V$ must hold. We state this as a lemma

[9]The set D may be empty.

for future reference. The lemma can be shown through the same logic as deriving (17) by letting $A^s = A'$, $B^s = V$, and $A^t = A''$

LEMMA 6. *Let $A', A'' \subset X$ and Γ be a consideration mapping satisfying SAFP. If both $\Gamma(A') \subset A' \backslash V$ and $A' \backslash V \subset A''$ hold for some $V \subset A'$, then $\Gamma(A'') \subset A'' \backslash V$.*

Now we turn to extending (18) with help of Lemma 6. We start from going one step further: consider the situation where for some $r, s, t \in \mathcal{T}$, it holds that $[(A^r \backslash B^r) \cup (A^s \backslash B^s)] \subset A^t$. Since $(A^r \backslash B^r) \cup (A^s \backslash B^s) = (A^r \cup A^s) \backslash (B^r \cap B^s)$, we have that $[(A^r \cup A^s) \backslash (B^r \cup B^s)] \subset [(A^r \cup A^s) \backslash (B^r \cap B^s)]$. In particular, both $(A^r \backslash B^r) \subset [(A^r \cup A^s) \backslash (B^r \cap B^s)]$ and $(A^s \backslash B^s) \subset [(A^r \cup A^s) \backslash (B^r \cap B^s)]$ hold. Then, applying Lemma 6 by letting $A' = A^r$, $A'' = [(A^r \cup A^s) \backslash (B^r \cap B^s)]$, and $V = B^r$, we have

$$\begin{aligned} \Gamma\left([(A^r \cup A^s) \backslash (B^r \cap B^s)]\right) &\subset [(A^r \cup A^s) \backslash (B^r \cap B^s)] \backslash B^r \\ &= (A^r \cup A^s) \backslash B^r. \end{aligned} \tag{19}$$

By applying Lemma 6 to A^s in an analogous way, we have

$$\begin{aligned} \Gamma\left([(A^r \cup A^s) \backslash (B^r \cap B^s)]\right) &\subset [(A^r \cup A^s) \backslash (B^r \cap B^s)] \backslash B^s \\ &= (A^r \cup A^s) \backslash B^s. \end{aligned} \tag{20}$$

Then, combining (19) and (20), it follows that

$$\Gamma\left([(A^r \cup A^s) \backslash (B^r \cap B^s)]\right) \subset [(A^r \cup A^s) \backslash (B^r \cup B^s)].$$

Since $[(A^r \cup A^s) \backslash (B^r \cup B^s)] \subset A^t$ follows from the definition, we can again apply Lemma 6 by letting $A' = (A^r \cup A^s) \backslash (B^r \cap B^s)$, $A'' = A^t$, and $V = B^r \cup B^t$ to conclude that

$$[(A^r \backslash B^r) \cup (A^s \backslash B^s)] \subset A^t \Longrightarrow \Gamma(A^t) \subset A^t \backslash (B^r \cup B^s). \tag{21}$$

Clearly, (21) is an extension of (17), and the former derives an extenstion of (18) such that

$$[(A^r \backslash B^r) \cup (A^s \backslash B^s)] \subset A^t \Longrightarrow a^t \notin B^r \cup B^s. \tag{22}$$

Then, by inductive argument, we can, in turn, extend (21) and (22) for any subset $\tau \subset \mathcal{T}$

such that $\left(\bigcup_{r\in\tau} A^r \setminus \bigcup_{r\in\tau} B^r\right) \subset A^t$. Namely, by the extension of (22), we have the following axiom as a necessary (and actually sufficient) condition for a data set $\mathcal{O}$ to be rationalizable by a substitutable limited attention model.

Axiom of Substitutable Limited Attention (ASLA): A data set $\mathcal{O} = \{(a^t, A^t)\}_{t\in\mathcal{T}}$ obeys the axiom of substitutable limited attention, if there exists a profile of cut-off points $\left[a^{t(q)}\right]_{q=1}^{Q}$ such that, for every $t \in \mathcal{T}$ and any set of indices $\tau \subset \mathcal{T}$,

$$\bigcup_{r\in\tau} A^r \Big\backslash \bigcup_{r\in\tau} B^r \subset A^t \Longrightarrow a^t \notin \bigcup_{r\in\tau} B^r. \tag{23}$$

Theorem 3. *A data set* $\mathcal{O} = \{(a^t, A^t)\}_{t\in\mathcal{T}}$ *is rationalizable by the substitutable limited attention model, if and only if it obeys ASLA.*

The substantial part of the proof is the sufficiency of ASLA. Similar to Theorems 1 and 2, we construct a pair of a consideration mapping and a strict preference that rationalizes $\mathcal{O}$. To define Γ, we need the following set of indices for every $A \subset X$:

$$\tau(A) = \max\left\{\tau \subset \mathcal{T} : \bigcup_{r\in\tau} A^r \Big\backslash \bigcup_{r\in\tau} B^r \subset A\right\}. \tag{24}$$

Then, by using $\tau(A)$, define Γ such that

$$\Gamma(A) = A \Big\backslash \bigcup_{r\in\tau(A)} B^r. \tag{25}$$

Obviously, in order for the above definition to be well-defined, $\tau(A)$ must be uniquely determined for every $A \subset X$, which is actually the case as proved in Appendix.

Lemma 7. *For every* $A \subset X$*,* $\tau(A)$ *is uniquely determined.*

Once we construct a consideration mapping as above, then the rest of proof follows a quite similar path to Theorems 1 and 2. The following two lemmas are proved in Appendix, and the proof completes by extending $\succ^*$ by using Szpilrajn's theorem.

Lemma 8. *The consideration mapping defined as (25) obeys SAFP.*

Lemma 9. *Let* $\succ^*$ *be a binary relation such that* $x'' \succ^* x'$*, if* $x'' = a^t$*,* $x' \in \Gamma(A^t)$*, and* $x'' \neq x'$*. Then* $\succ^*$ *is acyclic and for every* $t \in \mathcal{T}$*,* $a^t \in \Gamma(A^t)$*.*

Given Propositions 3, the following statement is immediate from Theorem 3.

Corollary 3. *The following statements are all equivalent.*

(a) A data set $\mathcal{O} = \{(a^t, A^t)\}_{t\in\mathcal{T}}$ obeys ASLA.

(b) A data set $\mathcal{O} = \{(a^t, A^t)\}_{t\in\mathcal{T}}$ is rationalizable by a substitutable limited attention model.

(c) A data set $\mathcal{O} = \{(a^t, A^t)\}_{t\in\mathcal{T}}$ is rationalizable by a substitutable rational menu choice model.

(d) A data set $\mathcal{O} = \{(a^t, A^t)\}_{t\in\mathcal{T}}$ is rationalizable by a substitutable complete rational menu choice model.

Finally, we point out that the joint of ALA and SUB does not work as a necessary and sufficient condition for a data set to be consistent with a substitutable limited attention model. In the example below, a data set obeys both ALA and SUB, and hence it is rationalizable respectively by a limited attention model and a substitutable consideration model. However, it fails to obey ASLA, or equivalently, it is not rationalizable by a substitutable limited attention model. This implies that, in general, the joint of two theoretical hypotheses is not necessarily tested by the joint of tests for each hypothesis.

Example 1. Let $X = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7\}$, and consider a data set of five observations as below:

t	1	2	3	4	5
A^t	$\{x_1, x_2, x_3\}$	$\{x_1, x_2, x_4, x_6\}$	$\{x_1, x_3, x_5, x_7\}$	$\{x_2, x_4, x_6\}$	$\{x_3, x_5, x_7\}$
a^t	x_1	x_2	x_3	x_4	x_5

There are four cycles with respect to the direct revealed preference $\succ^R$: $a^1 \succ^R a^2 \succ^R a^1$, $a^1 \succ^R a^3 \succ^R a^1$, $a^2 \succ^R a^4 \succ^R a^2$, and $a^3 \succ^R a^5 \succ^R a^3$. We claim that, by choosing (a^1, a^1, a^2, a^3) as a profile of cut-off points, this data set obeys both ALA and ASC, but violates ASLA. The relevant sets for this choice of cut-off points are summarized in Table 1. It is easily confirmed that both ALA and ASC are satisfied. Indeed, the latter is trivially satisfied, since we have $A^4 \subset A^2$ and $A^5 \subset A^3$, but B^4 and B^5 are both empty sets. It can also be seen that ALA is satisfied. Note that, for every s, t, we have $(A^s \backslash B^s) \cup (A^t \backslash B^t) \not\subset (A^s \cap A^t)$, and (13) is trivially satisfied. However, since $\{x_1\} = A^1 \backslash B^1 \subset A^2$ and $x_2 = a^2 \in B^1$, (18) is violated, let alone ASLA. In addition, as shown below, the profile of cut-off points (a^1, a^1, a^2, a^3) is the

q	1	2	3	4
$t(q)$	1	1	2	3
$A^{t(q)}$	$\{x_1, x_2, x_3\}$	$\{x_1, x_2, x_3\}$	$\{x_1, x_2, x_4, x_6\}$	$\{x_1, x_3, x_5, x_7\}$
$b^{t(q)}$	x_2	x_3	x_4	x_5

t	1	2	3	4	5
B^t	$\{x_2, x_3\}$	$\{x_4\}$	$\{x_5\}$	$\varnothing$	$\varnothing$
$A^t \backslash B^t$	$\{x_1\}$	$\{x_1, x_2, x_6\}$	$\{x_1, x_3, x_7\}$	$\{x_2, x_4, x_6\}$	$\{x_3, x_5, x_7\}$

Table 1: Relevant sets for cut-off points (a^1, a^1, a^2, a^3).

only profile that obeys both ALA and ASC. For a profile of cut-off points to satisfy ASC, it can contain neither a^4 nor a^5. To see this, suppose that $t(3) = 4$. Then we have $b^{t(3)} = x_2$, $B^{t(3)} = \{x_2\}$, $A^{t(3)} = A^4 \subset A^2$, and $a^2 = x_2 \in B^{t(3)}$, which violates ASC. Setting $t(4) = 5$ leads to a similar violation of ASC. Therefore, we must have $t(3) = 2$ and $t(4) = 3$ in the profile. Furthermore, if a profile of cut-off points satisfies ALA, it cannot have a^2 appear twice, or a^3 appear twice in the profile. To see this, consider profile (a^2, a^1, a^2, a^3). We have $B^2 = \{x_1, x_4\}$, and thus

$$\{x_2, x_6\} = A^2 \backslash B^2 \subset A^4 \subset A^2 = \{x_1, x_2, x_4, x_6\},$$

but $x_2 = a^2 \neq a^4 = x_4$, which is a violation of ALA. The case of profile (a^1, a^3, a^2, a^3) leads to a similar violation of ALA.[10]

4 Testing rational shortlisting models

By Proposition 4, if a data set $\mathcal{O} = \{(a^t, A^t)\}_{t \in \mathcal{T}}$ is collected from an agent obeying a (transitive) rational shortlisting model $(\Gamma, \succ)$, then it must obey ASC (ASLA). However, it is not diffcult to find a data set that obeys ASC (ASLA), but inconsistent with any (transitive) rational shortlisting model. Indeed, for a data set to be rationalizable by a rational shortlisting model, it must hold that for every $r, s, t \in \mathcal{T}$ with $A^r = A^s \cup A^t$,

$$a^s = a^t \Longrightarrow a^r = a^s = a^t, \tag{26}$$

[10] One can confirm that this example is consistent with the straightforward adaptation of LCA-WARP, a revealed preference characterization of a substitutable limited attention model for a choice function shown in Llears et al. (2010).

which is independent of ASC/ASLA.[11] In this section, we provide a test for the (transitive) rational shortlisting model.

Suppose that an agent has two preferences $\succ'$ and $\succ$, where the former is merely acyclic while the latter is a strict preference, and that a consideration mapping Γ is defined as (8). Similar to the previous models, we may assume that a data set $\mathcal{O}$ collected from such an agent contains cycles with respect to $\succ^R$, and let $\left[a^{t(q)}\right]_{q=1}^{Q}$ be a profile of cut-off points. Corresponding to this profile of cut-off points, the set B^t is determined as in (10) for every $t \in \mathcal{T}$. Recall, by the definition of cut-off points, for every $x' \in B^t$ we have $x' \notin \Gamma(A^t)$, which means that there exists some $x'' \in A^t \backslash x'$ such that $x'' \succ' x'$. On the other hand, $x' \in B^t$ means that x' is a chosen alternative in some observed feasible set, say A^s. Then, it must follow that $x' \nsucc^R x''$; otherwise, since we have $x'' \succ' x'$, the definition of Γ will require $x' \notin \Gamma(A^s)$, which contradicts that x' is the chosen alternative at A^s.

Given the discussion above, we can define a binary relation $\rhd$ on X such that: $x'' \rhd x'$ if $x' \in B^t$ for some $t \in \mathcal{T}$, $x'' \in A^t \backslash x'$, and $x' \nsucc^R x''$. Note that for every $x' \in B^t$, there exists at least one $x'' \in A^t \backslash x'$ with $x'' \rhd x'$ for which $x'' \succ' x'$ actually holds. Loosely speaking, $\rhd$ can be seen as a broad guess of the first step preference $\succ'$. In addition, the acyclicity of $\succ'$ requires that we can always find a selection $\rhd' \subset \rhd$ that is acyclic, and for every $t \in \mathcal{T}$ and $x' \in B^t$, there exists some $x'' \in A^t \backslash x'$ with $x'' \rhd' x'$. Furthermore, if the first step preference $\succ'$ is assumed to be transitive, a selection $\rhd'$ has to be chosen so that

$$\text{for every } x' \in B^t \text{ and } z^1, ..., z^k,\ x'' \rhd' z^1 \rhd' \cdots \rhd' z^k \rhd' x' \Longrightarrow x' \nsucc^R x''. \tag{27}$$

Loosely speaking, $\rhd'$ is a "correct" guess of the first step preference, and if transitivity is imposed, the above implies that $x'' \succ' x'$. Hence, if $x' \succ^R x''$ were to hold, then it leads a contradiction that x' is deleted from a consideration set from which is actually chosen. In fact, this observation, which is summarized in the axioms below, characterize a data set that is rationalizable by the (transitive) rational shortlisting model.

Axiom of Rational Shortlisting (ARS): A data set $\mathcal{O} = \{(a^t, A^t)\}_{t \in \mathcal{T}}$ obeys the axiom of rational shortlisting, if there exists a profile of cut-off points $\left[a^{t(q)}\right]_{q=1}^{Q}$ such that there exists

[11] If an agent obeys the rational shortlisting model, $\Gamma(A^r) \subset \Gamma(A^r) \cup \Gamma(A^t)$ is obvious. In addition, $x = a^t = a^s$ implies that no element in $A^t \cup A^r = A^r$ can dominate x with respect to the first step preference, and x dominates any other elements in $\Gamma(A^r) \subset \Gamma(A^r)$.

an acyclic selection $\rhd'$ of $\rhd$, where for every $t \in \mathcal{T}$,

$$\text{for every } x' \in B^t, \text{ there exists } x'' \in A^t \text{ with } x'' \rhd' x'. \tag{28}$$

Axiom of Transitive Rational Shortlisting (ATRS): A data set $\mathcal{O} = \{(a^t, A^t)\}_{t \in \mathcal{T}}$ obeys the axiom of transitive rational shortlisting, if there exists a profile of cut-off points $\left[a^{t(q)}\right]_{q=1}^{Q}$ such that there exists an acyclic selection $\rhd'$ of $\rhd$ that obeys (27) and (28).

Theorem 4. *A data set $\mathcal{O} = \{(a^t, A^t)\}_{t \in \mathcal{T}}$ is rationalizable by a rational shortlisting model, if and only if it obeys ARS.*

Theorem 5. *A data set $\mathcal{O} = \{(a^t, A^t)\}_{t \in \mathcal{T}}$ is rationalizable by a transitive rational shortlisting model, if and only if it obeys ATRS.*

Remark 1. By Proposition 4, it suffices to apply the above tests only to profiles of cut-off points that satisfy the requirement of ASC or ASLA, i.e. the condition (15) or (23).

The proofs of the above theorems are almost identical and the necessity parts of them have been already discussed. Hence, we only prove the sufficient parts of them. Given a profile of cut-off points that obeys (28), define Γ as

$$\Gamma(A) = \{x \in A : \nexists x' \in A \text{ such that } x' \rhd' x\}. \tag{29}$$

Note that the selection $\rhd'$ is acyclic, so we use it as a first step preference for the case of Theorem 4. If we can find $\rhd'$ so that it obeys (27) in addition to (28), then we use the transitive closure of it, say, $\rhd''$ as a first step preference and define Γ by using it instead of $\rhd'$. Note further that $\Gamma(A^t) \subset A^t \backslash B^t$, by the definition of $\rhd'$ and the construction of Γ. The remaining substantial parts of the proof are to show that $a^t \in \Gamma(A^t)$ for every $t \in \mathcal{T}$, and the binary relation $\succ^*$ defined as $x'' \succ^* x'$ if $x'' = a^t, x' \in \Gamma(A^t)$, and $x'' \neq x'$ is acyclic, which are proved in Appendix. Note that the following lemma is true even if Γ is defined by $\rhd''$ when ATRS is satisfied.

Lemma 10. *Let $\succ^*$ be a binary relation such that $x'' \succ^* x'$, if $x'' = a^t$ for some $t \in \mathcal{T}$, $x' \in \Gamma(A^t)$, and $x'' \neq x'$. Then $\succ^*$ is acyclic, and for every $t \in \mathcal{T}$, $a^t \in \Gamma(A^t)$.*

The rest of the proof is to extend the transitive closure of $\succ^*$ to a strict preference by using Szpilrajn's theorem. Then it can easily be seen that the data set is rationalized by a (transitive) rational shortlisting model $(\Gamma, \succ)$.

It is shown by Manzini and Mariotti (2007) that a rational shortlisting model can be characterized by a combination of two axioms on a data set, namely, WWARP and Expansion. The former is implied when the consideration mapping obeys SUB. The latter requires that for every $A', A'' \subset X$, if $x = f(A') = f(A'')$, then $x = f(A' \cup A'')$, where f is a choice function. Given this, one may be tempted to consider that a rational shortlisting model is tested by the joint of ASC and (26), a straightforward partial-observation version of Expansion. The following example shows that this is not the case, i.e. we present a data set that obeys ASC and (26), but violates ARS. A similar example can be found for the joint of ASLA and (26).

Example 2. Let $X = \{x_1, x_2, x_3, x_4, x_5, x_6\}$ and consider a data set of six observations as below:

t	1	2	3	4	5	6
A^t	$\{x_1, x_2\}$	$\{x_1, x_2, x_5\}$	$\{x_3, x_4\}$	$\{x_1, x_3, x_4\}$	$\{x_5, x_6\}$	$\{x_3, x_5, x_6\}$
a^t	x_1	x_2	x_3	x_4	x_5	x_6

It can be seen that Expansion is trivially satisfied, because the chosen alternatives are all different. Note that there are four cycles with respect to $\succ^R$: $a^1 \succ^R a^2 \succ^R a^1$, $a^3 \succ^R a^4 \succ^R a^3$, $a^5 \succ^R a^6 \succ^R a^5$, and $a^1 \succ^R a^2 \succ^R a^5 \succ^R a^6 \succ^R a^3 \succ^R a^4 \succ^R a^1$. We first show that ARS cannot be satisfied. Consider the cycle $a^1 \succ^R a^2 \succ^R a^1$. If we choose a^1 to be the cut-off point, we will have $a^2 = x_2 \in B^1$. However, then, there does not exist any $x \in A^1$ such that $x_2 \not\succ^R x$, and we cannot define $\rhd$ for x_2. Therefore, we need to choose a^2 as the cut-off point for this cycle. By the same logic, we must choose a^4 and a^6 to be the cut-off points of the second and third cycles respectively. Then we must have $x_5 \rhd x_1$, $x_1 \rhd x_3$, and $x_3 \rhd x_5$, and it will be impossible to find an acyclic selection of $\rhd$. This shows that ARS is violated.

Next we show that ASC is satisfied by considering the profile of cut-off points (a^2, a^4, a^6, a^4). Note that the only set inclusions of feasible sets that we have are $A^t \subset A^{t+1}$ for $t = 1, 3, 5$. On the other hand, since $B^t = \varnothing$ for $t = 1, 3, 5$, ASC is trivially satisfied.

5 Extension to panel data

In applied studies, it is often assumed that agents in a population can be partitioned into several *types*, and that agents with the same type share some common behavioral procedure. In particular, in the case of limited consideration model, one may consider the following two natural hypotheses of common behavioral procedures: one is assuming that agents with the same type have the same consideration mapping, and the other is assuming that agents with the same type have the same preference relation. To test such hypotheses, an observer may use a panel data set in the form of $\mathcal{O} = \left\{((a_i^t)_{i\in N}, A^t)\right\}_{t\in\mathcal{T}}$, with N being the set of agents who are thought to have a common consideration mapping/preference. Naturally, in both cases, restrictions from the model depend on the properties of a consideration mapping required in the model as shown in the subsequent subsections. In Sections 5.1 and 5.2, we extend the results of Section 3 to a panel data set, and consider the cases of common consideration mapping and preference respectively. In Section 5.3, we consider a panel data set version of Section 4.

5.1 Testing a common consideration

Suppose that a panel data set $\mathcal{O} = \left\{((a_i^t)_{i\in N}, A^t)\right\}_{t\in\mathcal{T}}$ is collected from agents all of whom obey a common limited consideration model. We say that a data set is rationalized by a specific *common limited consideration* model $(\Gamma, (\succ_i)_{i\in N})$ if for every $t \in \mathcal{T}$ and every $i \in N$, $a_i^t \in \Gamma(A^t)$ and $a_i^t \succ_i x$ for all $x \in \Gamma(A^t)$, where the consideration mapping Γ is common across agents. Considering the discussion in Section 3, for each agent $i \in N$, the individual data set $\mathcal{O}_i$ may have $Q_i(\geqslant 0)$ cycles with respect to the direct revealed preference $\succ_i^R$, and each cycle will have a cut-off point, where we denote the profile of such cut-off points by $\left[a_i^{t(q_i)}\right]_{q_i=1}^{Q_i}$. Define sets $B_i^t = \{b_i^{t(q_i)} \in A^t : a_i^{t(q_i)} = a_i^t\}$ in a parallel manner to Section 3. Then it follows that $\Gamma(A^t) \subset A^t \backslash B_i^t$, for all $i \in N$ and all $t \in \mathcal{T}$. Therefore, we can define the following set

$$B^t = \bigcup_{i\in N} B_i^t, \tag{30}$$

which is empty if there exists no agent i with cut-off point $a_i^{t(q_i)} = a_i^t$. As in Section 3, this set B^t plays crucial roles in revealed preference tests. Since for every $x \in B^t$, there exists an agent i where $x \succ_i a_i^t$, so we have $x \notin \Gamma(A^t)$. Put otherwise, for every $t \in \mathcal{T}$,

$$\Gamma(A^t) \subset A^t \backslash B^t. \tag{31}$$

Parallel to Section 3, the above constraint does not depend on the type of the model. On the other hand, since the set B^t depends on cut-off points of cycles of each agent, some restriction must be imposed in order for Γ to satisfy a specific structure like AFP, SUB, or SAFP. Since the form of the general principle (31) is identical to (11), the axioms that characterize each of the models can be derived in an identical manner. The only difference is that the choices can differ across agents.

ALA-C: A data set $\mathcal{O} = \{((a_i^t)_{i \in N}, A^t)\}_{t \in \mathcal{T}}$ obeys ALA-C, if for every $i \in N$, there exists a profile of cut-off points $\left[a_i^{t(q_i)}\right]_{q_i=1}^{Q_i}$ such that for every $s, t \in \mathcal{T}$,

$$(A^s \backslash B^s) \cup (A^t \backslash B^t) \subset (A^s \cap A^t) \Longrightarrow a_i^s = a_i^t \text{ for all } i \in N, \tag{32}$$

ASC-C: A data set $\mathcal{O} = \{((a_i^t)_{i \in N}, A^t)\}_{t \in \mathcal{T}}$ obeys ASC-C, if for every $i \in N$, there exists a profile of cut-off points $\left[a_i^{t(q_i)}\right]_{q_i=1}^{Q_i}$ such that for every $s, t \in \mathcal{T}$,

$$A^s \subset A^t \Longrightarrow a_i^t \notin B^s \text{ for all } i \in N. \tag{33}$$

ASLA-C: A data set $\mathcal{O} = \left\{((a_i^t)_{i \in N}, A^t)\right\}_{t \in \mathcal{T}}$ obeys ASLA-C, if for every $i \in N$, there exists a profile of cut-off points $\left[a_i^{t(q_i)}\right]_{q_i=1}^{Q_i}$ such that, for every $t \in \mathcal{T}$ and every set of indices $\tau \subset \mathcal{T}$,

$$\bigcup_{r \in \tau} A^r \Big\backslash \bigcup_{r \in \tau} B^r \subset A^t \Longrightarrow a_i^t \notin \bigcup_{r \in \tau} B^r \text{ for all } i \in N. \tag{34}$$

Proposition 5. *A data set $\mathcal{O} = \left\{((a_i^t)_{i \in N}, A^t)\right\}_{t \in \mathcal{T}}$ is rationalizable by a common limited attention model, if and only if it obeys ALA-C.*

Proposition 6. *A data set $\mathcal{O} = \{((a_i^t)_{i \in N}, A^t)\}_{t \in \mathcal{T}}$ is rationalizable by a common substitutable consideration model, if and only if it obeys ASC-C.*

Proposition 7. *A data set $\mathcal{O} = \left\{((a_i^t)_{i \in N}, A^t)\right\}_{t \in \mathcal{T}}$ is rationalizable by a common substitutable limited attention model, if and only if it obeys ASLA-C.*

The proofs of the propositions are almost identical to Theorems 1, 2, and 3 respectively. The construction of Γ is the same as (14), (16), and (25), and the lemmas that show AFP, SUB, and SAFP can be directly applied. The only difference is that we have to construct a strict preference for every agent. Defining, for each agent $i \in N$, a binary relation $\succ_i^*$ such that $x'' \succ_i^* x'$, if $x'' = a_i^t$ for some $t \in \mathcal{T}$, $x' \in \Gamma(A^t)$, and $x'' \neq x'$, Lemmas 3, 5, and 9 can be applied by replacing $\succ^*$ with $\succ_i^*$.

5.2 Testing a common preference

Considering a panel data set $\mathcal{O} = \left\{((a_i^t)_{i \in N}, A^t)\right\}_{t \in \mathcal{T}}$, we say that it is rationalized by a specific *limited consideration model with common preference* $((\Gamma_i)_{i \in N}, \succ)$, if for every $t \in \mathcal{T}$ and every $i \in N$, $a_i^t \in \Gamma_i(A^t)$, and $a_i^t \succ x$ for all $x \in \Gamma_i(A^t)$, where the consideration mappings may differ across agents, but all agents have the same preference.

Let $\succ^R$ be the union of the individual direct revealed preferences $\succ_i^R$, i.e. $\succ^R = \bigcup_{i \in N} \succ_i^R$. Note that, then, $x'' \succ^R x'$, if there exists some $i \in N$ and $t \in \mathcal{T}$ such that $a_i^t = x''$ and $x' \in A^t$. If $\succ^R$ does not generate any cycle, then a data set can be rationalized by the standard rational model with a common preference relation. Suppose that there are $Q (\geqslant 0)$ cycles with respect to $\succ^R$, where the q-th cycle is denoted as $(a^{t_k^q})_{k=1}^{K_q}$. Then there exists some profile of cut-off points $\left[a^{t(q)}\right]_{q=1}^{Q}$. Since all agents have the same preference, this profile does not include the agent index. However, the agents are allowed to have different consideration mappings, so we must consider the agent index when we consider agents' consideration mappings. Suppose that there exists some agent $i \in N$ such that $a_i^t = a^{t(q)}$ for some cycle q and $t \in \mathcal{T}$. Then since we have $b^{t(q)} \succ a^{t(q)} = a_i^t$, it must follow that $b^{t(q)} \notin \Gamma_i(A^t)$. Hence we define sets $B_i^t = \{b^{t(q)} \in A^t : a^{t(q)} = a_i^t\}$, for all $i \in N$ and $t \in \mathcal{T}$. Then $x \in B_i^t$ implies $x \succ a_i^t$, and thus $x \notin \Gamma_i(A^t)$. Put otherwise, for every $t \in \mathcal{T}$ and every $i \in N$,

$$\Gamma_i(A^t) \subset A^t \backslash B_i^t. \tag{35}$$

The discussion up to this point is a general result that is common among all the limited consideration models with common preference. In fact, the relations (11) and (35) are identical except for the agent index i. Thus, fixing agent $i \in N$, the discussion in this subsection will be completely parallel to Section 3, and we have the following axioms.

ALA-P: A data set $\mathcal{O} = \{((a_i^t)_{i\in N}, A^t)\}_{t\in\mathcal{T}}$ obeys ALA-P, if there exists a profile of cut-off points $\left[a^{t(q)}\right]_{q=1}^{Q}$ such that, for all agents $i \in N$, and for all $s, t \in \mathcal{T}$,

$$(A^s \backslash B_i^s) \cup (A^t \backslash B_i^t) \subset (A^s \cap A^t) \Longrightarrow a_i^s = a_i^t, \tag{36}$$

ASC-P: A data set $\mathcal{O} = \{((a_i^t)_{i\in N}, A^t)\}_{t\in\mathcal{T}}$ obeys ASC-P, if there exists a profile of cut-off points $\left[a^{t(q)}\right]_{q=1}^{Q}$ such that, for every agent $i \in N$ and every $s, t \in \mathcal{T}$,

$$A^s \subset A^t \Longrightarrow a_i^t \notin B_i^s. \tag{37}$$

ASLA-P: A data set $\mathcal{O} = \{((a_i^t)_{i\in N}, A^t)\}_{t\in\mathcal{T}}$ obeys ASLA-P, if there exists a profile of cut-off points $\left[a^{t(q)}\right]_{q=1}^{Q}$ such that, for all agents $i \in N$, every $t \in \mathcal{T}$, and every set of indices $\tau \subset \mathcal{T}$,

$$\bigcup_{r\in\tau} A^r \Big\backslash \bigcup_{r\in\tau} B_i^r \subset A^t \Longrightarrow a_i^t \notin \bigcup_{r\in\tau} B_i^r. \tag{38}$$

Proposition 8. *A data set $\mathcal{O} = \{((a_i^t)_{i\in N}, A^t)\}_{t\in\mathcal{T}}$ is rationalizable by the limited attention model with common preference, if and only if it obeys ALA-P.*

Proposition 9. *A data set $\mathcal{O} = \{((a_i^t)_{i\in N}, A^t)\}_{t\in\mathcal{T}}$ is rationalizable by the substitutable consideration model with common preference, if and only if it obeys ASC-P.*

Proposition 10. *A data set $\mathcal{O} = \{((a_i^t)_{i\in N}, A^t)\}_{t\in\mathcal{T}}$ is rationalizable by the substitutable limited attention model with common preference, if and only if it obeys ASLA-P.*

The proofs of the propositions are almost identical to Theorems 1, 2, and 3 respectively. However, since agents have different consideration mappings, there are slight differences. In constructing the consideration mappings, we must replace Γ with Γ_i, B^t with B_i^t, and $\tau(A)$ with $\tau_i(A)$, in (14), (16), and (25). Fixing agent $i \in N$, the lemmas that show AFP, SUB, and SAFP can be directly applied.

Care is needed in proving the acyclicity of binary relation $\succ^*$ defined such that $x'' \succ^* x'$, if $x'' = a_i^t$ for some $i \in N$ and $t \in \mathcal{T}$, $x' \in \Gamma_i(A^t)$, and $x'' \neq x'$. Suppose that there exists a cycle $a^{t_1} \succ^* a^{t_2} \succ^* \cdots \succ^* a^{t_L} \succ^* a^{t_1}$, and that the cut-off point is a^{t_ℓ}. Then, since all agents have different consideration, we must focus on every agent $i \in N$ with $a_i^{t_\ell} = a^{t_\ell}$, and show that $a^{t_{\ell+1}} \notin \Gamma_i(A^{t_\ell})$. However, the last part can be shown by fixing such $i \in N$, and apply the discussions in Lemmas 3, 5, and 9 respectively.

5.3 Rational shortlisting and panel data

The panel data discussion for (transitive) rational shortlisting models is simply a combination of the discussions in the preceding subsections and Section 4. As in the previous models, we consider the following two natural hypotheses of common behavioral procedures: (1) the agents have a common first step preference, and (2) the agents have a common second step preference.

We start from common first step preference models $(\Gamma, (\succ_i)_{i\in N})$. Note that under this assumption, all agents have a common consideration mapping Γ. As in Section 5.1, for each agent $i \in N$, the individual data set $\mathcal{O}_i$ may have $Q_i (\geqslant 0)$ cycles with respect to the direct revealed preference $\succ_i^R$, and we will have a profile of cut-off points $\left[a_i^{t(q_i)}\right]_{q_i=1}^{Q_i}$. Defining sets $B_i^t = \{b_i^{t(q_i)} \in A^t : a_i^{t(q_i)} = a_i^t\}$ and $B^t = \bigcup_{i\in N} B_i^t$, we have a general principle $\Gamma(A^t) \subset A^t \backslash B^t$, which is identical to (31). Note that for every $x' \in B^t$, there exists some $x'' \in A^t$ such that $x'' \succ' x'$ and $x' \not\succ_i^R x''$ for all $i \in N$. Thus we can define a binary relation $\rhd$ on X such that $x'' \rhd x'$, if $x' \in B^t$ for some $t \in \mathcal{T}$, $x'' \in A^t$, and $x' \not\succ_i^R x''$ for all $i \in N$. Then, similar to the case of Section 4, we have the following axioms.

ARS-F: A data set $\mathcal{O} = \{((a_i^t)_{i\in N}, A^t)\}_{t\in\mathcal{T}}$ obeys ARS-F, if for every $i \in N$ there exists a profile of cut-off points $\left[a_i^{t(q_i)}\right]_{q_i=1}^{Q_i}$, such that there exists an acyclic selection $\rhd'$ of $\rhd$ that obeys (28).

ATRS-F: A data set $\mathcal{O} = \{((a_i^t)_{i\in N}, A^t)\}_{t\in\mathcal{T}}$ obeys ATRS-F, if for every $i \in N$ there exists a profile of cut-off points $\left[a_i^{t(q_i)}\right]_{q_i=1}^{Q_i}$, such that there exists an acyclic selection $\rhd'$ of $\rhd$ that obeys (27) and (28).

Proposition 11. *A data set $\mathcal{O} = \{((a_i^t)_{i\in N}, A^t)\}_{t\in\mathcal{T}}$ is rationalizable by a common first step preference model, if and only if it obeys ARS-F.*

Proposition 12. *A data set $\mathcal{O} = \{((a_i^t)_{i\in N}, A^t)\}_{t\in\mathcal{T}}$ is rationalizable by a common transitive first step preference model, if and only if it obeys ATRS-F.*

The proofs of the propositions are almost identical to Theorems 4 and 5 respectively. The construction of Γ is the same as (29). The only difference is that we have to construct a strict second step preference for every agent. This can be done by replacing $\succ^*$ with $\succ_i^*$, where for every $i \in N$, $x'' \succ_i^* x'$, if $x'' = a_i^t$ for some $t \in \mathcal{T}$, $x' \in \Gamma(A^t)$, and $x'' \neq x'$.

Now, we turn to the case of common second step preference models $((\Gamma_i)_{i\in N}, \succ)$. Let $\succ^R$ be the union of the individual direct revealed preference relations $\succ_i^R$, and suppose that there are $Q(\geqslant 0)$ cycles with respect to $\succ^R$. Then there exists a profile of cut-off points $\left[a^{t(q)}\right]_{q=1}^{Q}$. Defining the sets $B_i^t = \{b^{t(q)} \in A^t : a^{t(q)} = a_i^t\}$, we have a general principle $\Gamma_i(A^t) \subset A^t \backslash B_i^t$, which is identical to (35). Note that for every agent $i \in N$, $x' \in B_i^t$ implies that there exists some $x'' \in A^t$ such that $x'' \succ_i' x'$ and $x' \not\succ_i^R x''$. Thus, for every $i \in N$, we can define a binary relation $\rhd_i$ on X such that $x'' \rhd_i x'$, if $x' \in B_i^t$ for some $t \in \mathcal{T}$, $x'' \in A^t$, and $x' \not\succ_i^R x''$. Then, similar to Section 4, we have the following axioms.

ARS-S: A data set $\mathcal{O} = \{((a_i^t)_{i\in N}, A^t)\}_{t\in\mathcal{T}}$ obeys ARS-S, if there exists a profile of cut-off points $\left[a^{t(q)}\right]_{q=1}^{Q}$ such that for every $i \in N$, there exists an acyclic selection $\rhd_i'$ of $\rhd_i$ that obeys (28).

ATRS-S: A data set $\mathcal{O} = \{((a_i^t)_{i\in N}, A^t)\}_{t\in\mathcal{T}}$ obeys ATRS-S, if there exists a profile of cut-off points $\left[a^{t(q)}\right]_{q=1}^{Q}$ such that for every $i \in N$, there exists an acyclic selection $\rhd_i'$ of $\rhd_i$ that obeys (27) and (28).

Proposition 13. *A data set $\mathcal{O} = \{((a_i^t)_{i\in N}, A^t)\}_{t\in\mathcal{T}}$ is rationalizable by a rational shortlisting model with common second step preference, if and only if it obeys ARS-S.*

Proposition 14. *A data set $\mathcal{O} = \{((a_i^t)_{i\in N}, A^t)\}_{t\in\mathcal{T}}$ is rationalizable by a transitive rational shortlisting model with common second step preference, if and only if it obeys ATRS-S.*

Again, the proofs of the propositions are almost identical to Theorems 4 and 5 respectively. However, since agents have different consideration mappings, we must replace Γ with Γ_i and $\rhd'$ with $\rhd_i'$ in (29).

Similar to the case of results in Section 5.2, care is needed in proving the acyclicity of binary relation $\succ^*$ defined such that $x'' \succ^* x'$, if $x'' = a_i^t$ for some $i \in N$ and $t \in \mathcal{T}$, $x' \in \Gamma_i(A^t)$, and $x'' \neq x'$. Suppose that there exists a cycle $a^{t_1} \succ^* a^{t_2} \succ^* \cdots \succ^* a^{t_L} \succ^* a^{t_1}$, and that the cut-off point is a^{t_ℓ}. Then, since all agents have different consideration, we must focus on every agent $i \in N$ with $a_i^{t_\ell} = a^{t_\ell}$, and show that $a^{t_{\ell+1}} \notin \Gamma_i(A^{t_\ell})$. However, the last part can be shown by fixing such $i \in N$, and apply the discussion in Lemma 10.

Appendix

Proof of Proposition 1

We first show that a rational menu choice model derives a limited attention model. Assume that there exists some asymmetric and transitive menu preference $\succ^M$, and that the consideration mapping Γ is defined as in (3). It suffices to show that Γ obeys AFP. Consider $A \in 2^X$ and $x \in A$ such that $x \notin \Gamma(A)$. By definition of Γ, this means that,

$$\text{for every set } B \subset A \text{ with } x \in B, \text{ there exists some } B' \subset A \text{ such that } B' \succ^M B. \tag{39}$$

Now we show that $\Gamma(A) = \Gamma(A \backslash x)$. First, take any $y \in \Gamma(A)$. This means that there exists some $B^* \subset A$ such that $y \in B^*$ and $B' \not\succ^M B^*$ for all $B' \subset A$. Note that, by (39), such B^* cannot contain x. Otherwise, we will end up with $x \in \Gamma(A)$, which contradicts our initial assumption. Therefore, it follows that $B^* \subset A \backslash x$, and $B' \not\succ^M B^*$ for all $B' \subset A \backslash x$. By definition of $\Gamma(A \backslash x)$, we conclude that $y \in \Gamma(A \backslash x)$, and we have $\Gamma(A) \subset \Gamma(A \backslash x)$. Next, take any $y \in \Gamma(A \backslash x)$. This means that there exists some $B^* \subset A \backslash x$ such that $y \in B^*$ and $B' \not\succ^M B^*$ for all $B' \subset A \backslash x$. Assume by way of contradiction that $y \notin \Gamma(A)$. Then there exists some $B^{**} \subset A$ such that $B^{**} \succ^M B^*$. Since $B' \not\succ^M B^*$ for all $B' \subset A \backslash x$, it follows that such B^{**} must contain x. On the other hand, since $x \notin \Gamma(A)$, there exists some $\bar{B} \subset A$ such that $\bar{B} \succ^M B^{**}$ and $x \notin \bar{B}$, i.e. $\bar{B} \subset A \backslash x$. Otherwise, since $\succ^M$ is asymmetric and transitive, there will be a menu $\bar{B}$ with $x \in \bar{B}$ that is maximal with respect to $\succ^M$, which contradicts $x \notin \Gamma(A)$. Then, since $\succ^M$ is asymmetric and transitive, we have $\bar{B} \succ^M B^*$, which contradicts $y \in \Gamma(A \backslash x)$. Summarizing, $y \in \Gamma(A)$, and thus $\Gamma(A \backslash x) \subset \Gamma(A)$. Since we have $\Gamma(A) = \Gamma(A \backslash x)$, we conclude that Γ obeys AFP.

Then, we show that a limited attention model can be converted into a complete rational menu choice model. Let a consideration mapping Γ obey AFP. It suffices to show that, given any $A \in 2^X$, $\Gamma(A)$ is defined as the optimal subset of A, with respect to some complete menu preference $\succ^M$. Note that this is equivalent to saying that the consideration mapping Γ obeys the weak axiom of revealed preference (WARP) on 2^X, that is, for $A', A'' \in 2^X$ with $A' \neq A''$,

$$\left[A', A'' \subset A \text{ and } A' = \Gamma(A)\right] \Longrightarrow \left[A'' \neq \Gamma(B) \text{ whenever } A', A'' \subset B\right]. \tag{40}$$

We prove that AFP of Γ implies WARP, by showing the contrapositive. Assume that Γ violates WARP, i.e. there exist $A', A'' \subset A \cap B$ such that $A' \neq A''$, $A' = \Gamma(A)$, and $A'' = \Gamma(B)$. Note that we have $\Gamma(A) \subset (A \cap B) \subset A$ and $\Gamma(B) \subset (A \cap B) \subset B$. Then, since $\Gamma(A) = A' \neq A'' = \Gamma(B)$, Γ cannot obey AFP. Summarizing, we conclude that the consideration mapping Γ in a limited attention model must obey WARP on 2^X, which means that there exists a complete, asymmetric, and transitive menu preference $\succ^M$ such that $\Gamma(A)$ is the optimal subset of A with respect to $\succ^M$. Hence, by using this $\succ^M$ as a menu preference, we can construct a complete rational menu choice model which has the same consideration mapping with the original limited attention model.

Finally, it is trivial to see that a complete rational menu choice model implies a rational menu choice model, since the former is a special case of the latter. □

Proof of Proposition 2

We first note that the equivalence of substitutable consideration models and order rationalization models has been shown by Cherepanov, Feddersen, and Sandroni (2013). Here we show that order rationalization models and categorize-then-choose models with a strict preference are observationally equivalent.

First we show that a categorize-then-choose model is a substitutable consideration model. Considering a categorize-then-choose model, an agent has a strict preference $\succ$ on X, and an asymmetric shading relation $\succ^M$ on 2^X. The consideration set for each $A \subset X$ is defined as in (7). It suffices to show that the corresponding consideration mapping Γ obeys SUB. To see this, consider $A', A'' \subset X$ such that $A' \subset A''$, and $x \in A'$ with $x \notin \Gamma(A')$, which means that there exist some $B', B'' \subset A'$ with $B'' \succ^M B'$ and $x \in B'$. Since we have $A' \subset A''$, it clearly follows that $B', B'' \subset A''$. By definition of the consideration mapping, we must have $x \notin \Gamma(A'')$.

Second, we show that a substitutable consideration model can be converted into a categorize-then-choose model. The proof will be constructive, i.e. we construct an asymmetric shading relation $\succ^M$, so that $\Gamma(A)$ is represented as in (7) for all $A \subset X$. Consider a substitutable consideration model $(\Gamma, \succ)$, and set $\succ^M$ on 2^X as follows: $A'' \succ^M A'$, if $A'' = \Gamma(A)$ and $A' = A \backslash \Gamma(A)$ for some $A \subset X$. By definition, $\succ^M$ is asymmetric. In what follows, we show that for every $A \subset X$, $\Gamma(A)$ is represented as in (7) by using $\succ^M$ as the shading relation.

Letting the set $\max(A, \succ^M)$ be such that

$$\max(A, \succ^M) = \{x \in A : \nexists B', B'' \subset A \text{ such that } B'' \succ^M B' \text{ and } x \in B'\},$$

it suffices to show that $\max(A, \succ^M) = \Gamma(A)$ for every $A \subset X$. We first show $\max(A, \succ^M) \subset \Gamma(A)$, by showing that $x \notin \Gamma(A)$ implies $x \notin \max(A, \succ^M)$. Note that by definition of the shading relation $\succ^M$, we have $\Gamma(A) \succ^M A \backslash \Gamma(A)$. Since $x \notin \Gamma(A)$, it follows that $x \notin \max(A, \succ^M)$. To show the opposite direction, take any $x \in \Gamma(A)$. Then, by letting f be the choice function corresponding to the substitutable consideration model (i.e. $f(A)$ is the most preferable element in $\Gamma(A)$), we have either $x = f(A)$ or $f(A) \succ x$. In the former case, to show that $x \in \max(A, \succ^M)$, suppose to the contrary. Then there exist some $B', B'' \subset A$ such that $B'' \succ^M B'$ and $x \in B'$. Defining $B = B' \cup B''$, it follows from the construction of $\succ^M$ that $B'' = \Gamma(B)$ and $B' = B \backslash \Gamma(B)$. Note that we have $B \subset A$, and SUB of Γ requires $\Gamma(A) \cap B \subset \Gamma(B)$. Then since $x = f(A) \in (\Gamma(A) \cap B)$, it must follow that $x \in \Gamma(B) = B''$, which contradicts $x \in B'$. Then, consider the case where $x \in \Gamma(A)$ and $f(A) \succ x$, and suppose by way of contradiction that $x \notin \max(A, \succ^M)$. This means that there exist some $B', B'' \subset A$ such that $B'' \succ^M B'$ and $x \in B'$. Defining, again, $B = B' \cup B''$, it follows from the definition of $\succ^M$ that $B'' = \Gamma(B)$ and $B' = B \backslash \Gamma(B)$. However, this is a contradiction, since $x \in \Gamma(A)$ and SUB of Γ requires $x \in \Gamma(B) = B''$. □

Proof of Lemma 1

Suppose that for some $s, t \in \mathcal{T}$ both $A^t \backslash B^t \subset A \subset A^t$ and $A^s \backslash B^s \subset A \subset A^s$ simultaneously hold. Then, it follows that $(A^t \backslash B^t) \cup (A^s \backslash B^s) \subset (A^t \cap A^s)$. By ALA, we must have $a^t = a^s$, and then $B^t \cap A = B^s \cap A$ follows from the assumption that $A \subset (A^t \cap A^s)$ and the definition of these sets. Thus we conclude that $A \backslash B^t = A \backslash B^s$. □

Proof of Lemma 2

We show that Γ as defined in (14) obeys AFP. Consider $A \in 2^X$ and $x \in A$ such that $x \notin \Gamma(A)$. This implies that there exists some t such that $A^t \backslash B^t \subset A \subset A^t$ and $x \in B^t$. Note that, by definition, $\Gamma(A) = A \backslash B^t$. Now consider the set $A \backslash x$. Since $x \in B^t$, it follows that $A^t \backslash B^t \subset A \backslash x \subset A^t$, and thus $\Gamma(A \backslash x) = (A \backslash x) \backslash B^t$. Recalling that $x \in B^t$, it follows that

$\Gamma(A\backslash x) = (A\backslash x)\backslash B^t = A\backslash B^t = \Gamma(A)$. □

Proof of Lemma 3

To see that $\succ^*$ is acyclic, suppose to the contrary, that is, there exists a cycle with respect to $\succ^*$, expressed as: $a^{t_1} \succ^* a^{t_2} \succ^* \cdots \succ^* a^{t_L} \succ^* a^{t_1}$. Note that $x'' \succ^* x'$ implies $x'' \succ^R x'$, which follows by the way these binary relations are defined. Therefore, the cycle above implies $a^{t_1} \succ^R a^{t_2} \succ^R \cdots \succ^R a^{t_L} \succ^R a^{t_1}$. Then, there must exist some cut-off point a^{t_ℓ}. Let us denote $t_\ell = t(q)$ and $a^{t_{\ell+1}} = b^{t(q)}$ for some $q \in \{1, \ldots, Q\}$. Note that $a^{t_{\ell+1}} = b^{t(q)} \in B^{t_\ell}$, and it follows by (14) that $a^{t_{\ell+1}} \notin \Gamma(A^{t_\ell})$. Hence it is impossible to have $a^{t_\ell} \succ^* a^{t_{\ell+1}}$, and we conclude that a cycle as above cannot exist.

The fact that $a^t \in \Gamma(A^t)$ for every $t \in \mathcal{T}$ follows directly from AFP. To see this, suppose not. Then there exists some $s \in \mathcal{T}$ such that $A^s\backslash B^s \subset A^t \subset A^s$ and $a^t \in B^s$. However, this is impossible, since AFP requires $a^t = a^s$, which contradicts $a^t \in B^s$. Summarizing, we have shown that $\succ^*$ is acyclic and a^t maximizes $\succ^*$ within the set $\Gamma(A^t)$ for every $t \in \mathcal{T}$. □

Proof of Lemma 4

We show that Γ obeys SUB. Consider $A', A'' \in 2^X$ such that $A' \subset A''$, and $x \in A'$ with $x \notin \Gamma(A')$. Then it suffices to show $x \notin \Gamma(A'')$. Note that $x \notin \Gamma(A')$ implies that there exist some t such that $A^t \subset A'$ and $x \in B^t$. Since $A' \subset A''$, we clearly have $A^t \subset A''$, and it follows that $x \notin \Gamma(A'')$. □

Proof of Lemma 5

To see that $\succ^*$ is acyclic, suppose to the contrary, that is, there exists a cycle with respect to $\succ^*$, expressed as: $a^{t_1} \succ^* a^{t_2} \succ^* \cdots \succ^* a^{t_L} \succ^* a^{t_1}$. Note that $x'' \succ^* x'$ implies $x'' \succ^R x'$, which follows by the way these binary relations are defined. Therefore, the cycle above implies $a^{t_1} \succ^R a^{t_2} \succ^R \cdots \succ^R a^{t_L} \succ^R a^{t_1}$. Then, there must exist some cut-off point a^{t_ℓ}. Let us denote $t_\ell = t(q)$ and $a^{t_{\ell+1}} = b^{t(q)}$ for some $q \in \{1, \ldots, Q\}$. Note that $a^{t_{\ell+1}} = b^{t(q)} \in B^{t_\ell}$, and it follows by (16) that $a^{t_{\ell+1}} \notin \Gamma(A^{t_\ell})$. Hence it is impossible to have $a^{t_\ell} \succ^* a^{t_{\ell+1}}$, and we conclude that a cycle as above cannot exist.

Next, we show that for every $t \in \mathcal{T}$, $a^t \in \Gamma(A^t)$. Suppose not. Then there exists some $s \in \mathcal{T}$ such that $A^s \subset A^t$ and $a^t \in B^s$. However, this is impossible, since ASC requires that $a^t \notin B^s$.

Summarizing, we have shown that $\succ^*$ is acyclic and a^t maximizes $\succ^*$ within the set $\Gamma(A^t)$ for every $t \in \mathcal{T}$. $\square$

Proof of Lemma 7

Suppose by way of contradiction that $\tau(A)$ is not unique, i.e. there exist $\tau_1(A) \neq \tau_2(A)$ that obey (24). Then $\left(\bigcup_{r\in\tau_1(A)} A^r \setminus \bigcup_{r\in\tau_1(A)} B^r\right) \subset A$ and $\left(\bigcup_{r\in\tau_2(A)} A^r \setminus \bigcup_{r\in\tau_2(A)} B^r\right) \subset A$. Hence $\left(\bigcup_{r\in\tau_1(A)} A^r \setminus \bigcup_{r\in\tau_1(A)} B^r\right) \cup \left(\bigcup_{r\in\tau_2(A)} A^r \setminus \bigcup_{r\in\tau_2(A)} B^r\right) \subset A$, which can be expressed as $\left[\bigcup_{r\in\tau_1(A)\cup\tau_2(A)} A^r \setminus \left(\bigcup_{r\in\tau_1(A)} B^r \cup \bigcup_{r\in\tau_2(A)} B^r\right)\right] \subset A$. Then, this implies

$$\left[\bigcup_{r\in\tau_1(A)\cup\tau_2(A)} A^r \Big\backslash \bigcup_{r\in\tau_1(A)\cup\tau_2(A)} B^r\right] \subset A.$$

By defining $\tau(A) = \tau_1(A) \cup \tau_2(A)$, we have $\tau(A) \supsetneq \tau_i(A)$ for $i = 1, 2$, which contradicts the maximality of $\tau_1(A)$ and $\tau_2(A)$. $\square$

Proof of Lemma 8

To see that Γ obeys SUB, consider $A', A'' \subset X$ with $A' \subset A''$, and $x \in A'$ such that $x \notin \Gamma(A')$. This means that $x \in \bigcup_{r\in\tau(A')} B^r$. Since $\tau(\cdot)$ is clearly monotonic, it follows that $\tau(A') \subset \tau(A'')$, and hence, $x \in \bigcup_{r\in\tau(A'')} B^r$. This assures that $x \notin \Gamma(A'')$.

To see that Γ obeys AFP, take any $A \subset X$ and any $x \in A$ with $x \notin \Gamma(A)$. This means that $x \in \bigcup_{r\in\tau(A)} B^r$, which in turn implies that

$$\left(\bigcup_{r\in\tau(A)} A^r \Big\backslash \bigcup_{r\in\tau(A)} B^r\right) \subset A\backslash x. \tag{41}$$

The maximality and uniqueness of $\tau(\cdot)$, combined with (41), imply $\tau(A) \subset \tau(A\backslash x)$. On the other hand, the monotonicity of $\tau(\cdot)$ implies $\tau(A\backslash x) \subset \tau(A)$. Hence we have $\tau(A) = \tau(A\backslash x)$. Then, we have $\Gamma(A\backslash x) = (A\backslash x)\backslash \bigcup_{r\in\tau(A\backslash x)} B^r = A\backslash \bigcup_{r\in\tau(A)} B^r = \Gamma(A)$. $\square$

Proof of Lemma 9

To see that $\succ^*$ is acyclic, suppose to the contrary, that is, there exists a cycle with respect to $\succ^*$, expressed as: $a^{t_1} \succ^* a^{t_2} \succ^* \cdots \succ^* a^{t_L} \succ^* a^{t_1}$. Note that $x'' \succ^* x'$ implies $x'' \succ^R x'$, which follows by the way these binary relations are defined. Therefore, the cycle above implies $a^{t_1} \succ^R a^{t_2} \succ^R \cdots \succ^R a^{t_L} \succ^R a^{t_1}$. Then, there must exist some cut-off point a^{t_ℓ}. Let us denote $t_\ell = t(q)$ and $a^{t_{\ell+1}} = b^{t(q)}$ for some $q \in \{1, \ldots, Q\}$. Note that $A^{t(q)} \backslash B^{t(q)} \subset A^{t(q)}$ holds, which implies $t(q) \in \tau(A^{t(q)})$, which in turn implies that $a^{t_{\ell+1}} = b^{t(q)} \in \bigcup_{r \in \tau(A^{t(q)})} B^r$, and $a^{t_{\ell+1}} \notin \Gamma(A^{t_\ell})$ follows. Thus it is impossible to have $a^{t_\ell} \succ^* a^{t_{\ell+1}}$.

Next, we show that for every $t \in \mathcal{T}$, $a^t \in \Gamma(A^t)$. In fact, this follows immediately from ASLA. For every $t \in \mathcal{T}$, we have $\left(\bigcup_{r \in \tau(A^t)} A^r \backslash \bigcup_{r \in \tau(A^t)} B^r\right) \subset A^t$. Then, ASLA requires $a^t \notin \bigcup_{r \in \tau(A^t)} B^r$. Recalling the definition of Γ in (25), we have $a^t \in \Gamma(A)$ for every $t \in \mathcal{T}$. □

Proof of Lemma 10

To prove that $\succ^*$ is acyclic, suppose to the contrary, i.e. there is a cycle: $a^{t_1} \succ^* a^{t_2} \succ^* \cdots \succ^* a^{t_L} \succ^* a^{t_1}$. Since we have $\succ^* \subset \succ^R$, this cycle implies $a^{t_1} \succ^R a^{t_2} \succ^R \cdots \succ^R a^{t_L} \succ^R a^{t_1}$. Then there must exist a cut-off point a^{t_ℓ}, and we have $a^{t_{\ell+1}} \in B^{t_\ell}$. By ARS, there exists some $x \in A^{t_\ell}$ such that $x \rhd' a^{t_{\ell+1}}$, which in turn implies $a^{t_{\ell+1}} \notin \Gamma(A^{t_\ell})$. Then it is impossible to have $a^{t_\ell} \succ^* a^{t_{\ell+1}}$, and we conclude that $\succ^*$ is acyclic.

Now we show that $a^t \in \Gamma(A^t)$ for every $t \in \mathcal{T}$. Assume that a data set obeys ARS and Γ is defined as the set of maximal elements with respect to $\rhd'$. By way of contradiction, suppose that for some $t \in \mathcal{T}$, $a^t \notin \Gamma(A^t)$. This means that there exists $x \in A^t \backslash a^t$ such that $x \rhd' a^t$, which in turn implies $x \rhd a^t$. However, this is not possible, since $x \rhd a^t$ requires $a^t \not\succ^R x$, while we have $a^t \succ^R x$. When a data set obeys ATRS and Γ is defined as the set of maximal elements with respect to $\rhd''$, $a^t \notin \Gamma(A^t)$ implies the existence of some $x \in A^t \backslash a^t$ such that $x \rhd'' a^t$. However, this is also impossible, since $x \rhd'' a^t$ implies the existence of a sequence $z^1, z^2, ..., z^k$ such that $x \rhd' z^1 \rhd' \cdots \rhd' z^k \rhd' a^t$, and by ATRS, $a^t \not\succ^R x$, which contradicts the assumption that $x \in A^t$. □

References

[1] Au, P.H. and K. Kawai, (2011): Sequentially rationalizable choice with transitive rationales. *Games and Economic Behavior*, 73, 608-614.

[2] Cherepanov, V., T. Feddersen, and A. Sandroni, (2013): Rationalization. *Theoretical Economics*, 8, 775-800.

[3] de Clippel, G. and K. Rozen, (2014): Bounded rationality and limited datasets. Mimeo.

[4] Lleras, J.S., Y. Masatlioglu, D. Nakajima, and E. Ozbay, (2010): When more is less: limited consideration. Mimeo.

[5] Loomes, G., C. Starmer, and R. Sugden, (1991): Observing violations of transitivity by experimental methods. *Econometrica*, 59, 425-439.

[6] Manzini, P. and M. Mariotti, (2007): Sequentially rationalizable choice. *The American Economic Review*, 97, 1824-1839.

[7] Manzini, P. and M. Mariotti, (2012): Categorize then choose: boundedly rational choice and welfare. *Journal of the European Economic Association*, 10, 1141-1165.

[8] Masatlioglu, Y., D. Nakajima, and E.Y. Ozbay, (2012): Revealed attention. *American Economic Review*, 102, 2183-2205.

[9] Tversky, A. (1969): Intransitivity of preferences. *Psychological Review*, 76, 31-48.

[10] Tyson, C.J. (2013): Behavioral implications of shortlisting procedures. *Social Choice and Welfare*, 41, 941-963.